Lecture 3

[Etingof §2.3.1]

A **representation** of an algebra A (also called a **left A-module**) is a vector space V with an algebra map $\rho : A \to End(V)$.

A **right A-module** is a vector space V with an anti-homomorphism $\rho : A \to End(V)$ i.e. $\rho(ab) = \rho(b)\rho(a)$, $\rho(1) = 1$.

Notation For left A-modules we abbreviate $\rho(a)v$ by $a.v$ or av and for right A-modules we write $v.a$ or va for $\rho(a)v$. Then

$$a.(b.v) = (ab).v \quad \text{for left } A\text{-mods}$$

$$(v.a).b = v.(ab) \quad \text{for right}$$

If A is commutative, any left A-module becomes a right A-module by $v.a := av$.

Ex. 1) $V = 0$

2) $V = A$, $a \cdot b = ab$ (product in A)

3) $A = k$ then a rep of A is just a vector space

4) $A = k\langle x_1, \dots, x_n \rangle$ then a rep of A is uniquely determined by specifying any n linear operators $\rho(x_1), \dots, \rho(x_n)$.

Def A subrepresentation of a rep. V of A is a subspace $U \subseteq V$ such that $\rho(a)U \subseteq U \ \forall a \in A$

Ex. $0 = \{0\}$ and V

Def A rep $V \neq 0$ of A is irreducible if the only subreps of V are 0 and V.

Def Let V_1, V_2 be reps of an alg A. A **homomorphism** (or **intertwining operator**) $\phi: V_1 \to V_2$ is a linear map such that

$$\phi \circ \rho_1(a) = \rho_2(a) \circ \phi \quad \forall a \in A$$

equivalently $\phi(av) = a\phi(v)$ $\forall a \in A \; \forall v \in V_1$

or

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ \downarrow \rho_1(a) & & \downarrow \rho_2(a) \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

The space of all homomorphisms $V_1 \to V_2$ is denoted:

$$\mathrm{Hom}_A(V_1, V_2).$$

If ϕ invertible then ϕ^{-1} is also a homomorphism, and ϕ is an **isomorphism**.

V_1 V_2 are **equivalent** or **isomorphic** if $\exists$ isomorphism $\phi: V_1 \to V_2$.

Def V_1, V_2 reps of A

then $V_1 \oplus V_2$ ← direct sum is a rep of A

with $\rho(a) = \left[\begin{array}{c|c} \rho_1(a) & 0 \\ \hline 0 & \rho_2(a) \end{array}\right]$

equivalently, with

$V_1 \oplus V_2 = \{(v, w) \mid v \in V_1, w \in V_2\}$

$a \cdot (v_1, v_2) = (a \cdot v_1, a \cdot v_2)$.

Def A rep V of A is

— indecomposable if V is not isomorphic to a direct sum of nonzero subreps.

Every irred. rep is indec.

but not conversely!

Schur's Lemma Let V_1, V_2 be reps of an alg A.
Let $\phi : V_1 \to V_2$ be a nonzero intertwining operator
Then

1) If V_1 is irreducible, ϕ is injective

2) If V_2 is irreducible, ϕ is surjective

So if both V_i are irr, then ϕ is an isomorphism.

pf (see book)

Lecture 4

[Etingof §2.3, cont'd.]

Cor (Schur's Lemma for $\overline{\Bbbk} = \Bbbk$)

Let V be a fin. dim'l irrep of A with $\overline{\Bbbk} = \Bbbk$. Let $\phi : V \to V$ be intertwining op. Then

$$\phi = \xi \mathrm{Id}_V$$

for some $\xi \in \Bbbk$.

Pf ξ eigenvalue of ϕ.

Then $\phi - \xi \mathrm{Id} : V \to V$ not iso. So $= 0$.

Cor. A comm alg / $\overline{\Bbbk} = \Bbbk$. Then every irrep is 1-dim'l

EX 2.3.14

1) $A = k$

The only indec rep is $V = k$

2) $A = k[x]$ $\quad \bar{k} = k$

Irreps: $V_\lambda = k$ with $x \cdot 1 = \lambda$ $(\lambda \in k)$

Indecs: Jordan blocks, $\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$

Note: Not every indec rep is irr.

3) $A = k[G]$ group alg

$\underline{\text{Rep}}\, A = \underline{\text{Rep}}\, G$

Def A rep of a group G is a v.sp. V with a grp hom $\rho : G \to GL(V)$

$$\underline{\text{Rep}}\, A \xrightarrow{\sim} \text{Rep}\, G$$

$$\rho_A \longmapsto \rho_A|_G$$

§2.4 IDEALS

Def A subspace I of an algebra A is a

Left ideal if $a \cdot x \in I \quad \forall a \in A, x \in I$

right ideal if $x \cdot a \in I \quad \forall a \in A, x \in I$

two-sided ideal if it's both a left ideal and a right ideal.

Note Left ideals are the same thing as subrepresentations of the regular rep.

Any right ideal in A is a left ideal in A^{op} ← the opposite alg

$$a \cdot_{op} b := ba$$

$$m_{A^{op}}(a \otimes b) = (m_A \circ \tau)(a \otimes b)$$ ← flip map

$$m_{A^{op}} = m_A \circ \tau$$

Ex 1) 0, A two-sided ideals

An alg is simple if these are the only two-sided ideals (and $0_A \neq 1_A$)

2) If $\phi : A \to B$ alg map then $\ker \phi$ is a two-sided ideal of A.

3) If $S \subseteq A$ is any subset the two-sided ideal generated by S, denoted $\langle S \rangle$, is

$$\langle S \rangle := \mathrm{Span}_k \{ asb \mid a, b \in A, s \in S \}$$

Similarly we can define left and right versions:

$$\langle S \rangle_l = \mathrm{Span} \{ as \mid a \in A, s \in S \}$$

$$\langle S \rangle_r = \mathrm{Span} \{ sa \mid s \in S, a \in A \}$$

$I \subset A$ maximal (left/right/2-sided) ideal

if 1) $I \neq A$

2) $I \subseteq J \subsetneq A \Rightarrow I = J$

Fact TFAE

1) every alg has a maximal ideal

2) Axiom of Choice holds

§2.5 Quotients

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§2.6 Gen & Rels.

$$\frac{k\langle x_1 \cdots x_n \rangle}{I}$$

$I = \langle S \rangle$

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§2.7

Ex Weyl alg (basis)

using $\hookrightarrow \mathrm{End}_k(k[x])$

q-Weyl alg

$U(sl_2)$

ρ faithful if inj

Lecture 5

Examples of Algebras [Etingof §2.7]

1) The Weyl algebra is

$$A_1(k) = k\langle x, y\rangle / \langle yx - xy - 1\rangle$$

2) The q-Weyl algebra ($q \in k^\times = k \setminus \{0\}$) is

$$A_1^q(k) = k\langle x, x^{-1}, y, y^{-1}\rangle / I$$

where I is generated by

$$yx - qxy$$

and the "obvious" elements

$$xx^{-1} - x^{-1}x \,, \quad xx^{-1} - 1 \,,$$
$$yy^{-1} - y^{-1}y \,, \quad yy^{-1} - 1$$

3) The enveloping algebra of $\mathfrak{sl}_2$ is

$$U(\mathfrak{sl}_2) = k\langle e, f, h\rangle \Big/ \left\langle \begin{array}{c} he - eh - 2e \\ hf - fh + 2f \\ ef - fe - h \end{array} \right\rangle$$

4) The n:th Weyl algebra is

$$A_n(k) = \underbrace{A_1(k) \otimes \cdots \otimes A_1(k)}_{n}$$

Thm $\{x^i y^j \mid i,j \in \mathbb{Z}_{\geq 0}\}$ is a basis for $A_1(k)$

Proof 1) $[y, x^k] = k x^{k-1}$, $k \geq 1$

induction on k: ($k=1$ trivial)

We will use the product rule

$[a, bc] = [a,b]c + b[a,c]$ (check!)

We have

$$[y, x^k] = [y,x]x^{k-1} + x[y, x^{k-1}]$$
$$= 1 \cdot x^{k-1} + x(k-1)x^{k-2}$$
$$= kx^{k-1}$$

↑ ind. hyp.

2) $A_1(k) = \text{span}_k \{x^i y^j \mid i,j \in \mathbb{Z}_{\geq 0}\}$

Since x, y generate $A_1(k)$ and belongs to RHS, it suffices to show RHS is a subalgebra of $A_1(k)$. Sufficen to show

$$x^i y^j x^k y^\ell \in \text{RHS}$$

For this, it suffices to show $y^j x^k \in$ RHS. We have (if $j > 0$):

$$y^j x^k = y^{j-1} y x^k = y^{j-1}(x^k y) + [y, x^k] \in \text{RHS}$$

by 1) and by induction on j.

3) Linear independence.
We define

$$\varphi : \{x, y\} \longrightarrow \mathrm{End}_k(k[x])$$

by

$$\varphi(x)\, p(x) = x \cdot p(x)$$
$$\varphi(y)\, p(x) = p'(x) = \frac{d}{dx} p(x)$$

By universal property of $k\langle x, y\rangle$, φ extends to an alg map

$$\tilde{\varphi} : k\langle x, y\rangle \longrightarrow \mathrm{End}_k(k[x])$$

We check $\langle yx - xy - 1 \rangle \subset \ker \tilde{\varphi}$
Since $\ker \tilde{\varphi}$ is an ideal, it suffices to check $yx - xy - 1 \in \ker \tilde{\varphi}$.
We have

$$\tilde{\varphi}(yx - xy - 1)(p(x)) =$$
$$= \left(\tilde{\varphi}(y)\tilde{\varphi}(x) - \tilde{\varphi}(x)\tilde{\varphi}(y) - \tilde{\varphi}(1)\right)(p(x))$$
$$= \left(\varphi(y)\varphi(x) - \varphi(x)\varphi(y) - \mathrm{id}\right)(p(x))$$
$$= \frac{d}{dx}\left(x p(x)\right) - x \cdot \frac{d}{dx}(p(x)) - p(x)$$
$$= \cancel{1 \cdot p(x)} + \cancel{x \frac{d}{dx}(p(x))} - \cancel{x \frac{d}{dx}(p(x))} - \cancel{p(x)} = 0$$

└ product rule for formal d/dx

The point: It suffices to show

$$\{\tilde{\varphi}(x^i y^j) \mid i,j \geq 0\}$$

is lin indep in $\mathrm{End}_k(k[x])$.

$$\tilde{\varphi}(x^i y^j) = x^i \frac{d^j}{dx^j}$$

Suppose

$$\sum_{i,j} \lambda_{ij} x^i \left(\frac{d}{dx}\right)^j = 0$$

We can write it as

$$\sum_j Q_j(x) \left(\frac{d}{dx}\right)^j = 0 \quad (*)$$

where $Q_j(x) \in k[x]$.

Apply to $\underline{1}$:

$$0 = \sum_j Q_j(x) \left(\frac{d}{dx}\right)^j (\underline{1}) = Q_0(x)$$

Apply instead to x

$$0 = \sum_{j \geq 1} Q_j(x) \left(\frac{d}{dx}\right)^j (x) = Q_1(x)$$

$$\cdots 0 = \sum_{j \geq l} Q_j(x) \left(\frac{d}{dx}\right)^j (x^l) = l! Q_l(x)$$

Assuming $\operatorname{char} k = 0$, we can cancel $\ell!$ and get

$$Q_\ell(x) = 0 \quad \forall \ell.$$

∎

§2.9 Lie Algebras.

Def - - -

Ex 2.9.2.

1) abelian

2) assoc alg, $\mathfrak{gl}(V)$

3) $U \subset A$, $[U, U] \subset U$
subspace assoc alg

4) $\mathrm{Der}\, A$

5) $\mathfrak{a} \subset \mathfrak{g}$ $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$

$$U(\mathfrak{g}) = T(\mathfrak{g}) \Big/ \left\langle \{ x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g} \} \right\rangle$$

Remark Ado's Thm says any fd Lie alg is a Lie subalg of $gl(V)$ for some f.d. V.

Remark Derivations are infinitesimal automorphisms

Ex $\mathbb{R}^3$, $\times$

$\mathfrak{sl}(n)$, $\mathfrak{sl}(2)$

$\mathfrak{H} = \begin{bmatrix} 0 & * & * \\ & 0 & * \\ & & 0 \end{bmatrix}$

$\mathfrak{aff}(1) = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$

$\mathfrak{so}(n)$ $\quad A^T = -A$

Def — A homomorphism $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras is a linear map s.t.

$$\varphi([xy]) = [\varphi(x)\varphi(y)]$$

A rep of a Lie alg $\mathfrak{g}$ is a V.sp. with

$$\rho: \mathfrak{g} \to \mathfrak{gl}(V) = \mathcal{L}(\mathrm{End}\, V)$$

(a Lie alg homomorphism)

Ex 1) $V = 0$

2) Any V with $\rho(x) = 0 \ \forall x \in \mathfrak{g}$

3) $V = \mathfrak{g}$, $\rho(x)(y) = [x, y]$

adjoint rep. One meaning of the Jacobi Identity is that ρ is a homomorphism.

Def $U(\mathfrak{g})$ basis-dependent version:

$\{x_i\}$ basis with $[x_i, x_j] = \sum_k c_{ij}^k x_k$

(c_{ij}^k: Structure Constants)

Then $U(\mathfrak{g}) := k\langle \{x_i\} \rangle / \text{Rels}$

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$$

Ex $U(\mathfrak{sl}_2)$ $\mathfrak{sl}_2$ has basis

$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = e^T$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

These satisfy

$$[ef]=h \quad [he]=2e \quad [h,f]=-2f$$

(check!) so

$$\mathfrak{u}(\mathfrak{sl}_2) = \frac{k\langle e,f,h\rangle}{\left(\begin{array}{l} ef-fe=h \\ he-eh=2e \\ hf-fh=-2f \end{array}\right)}$$

Lie groups are groups which are also smooth manifolds, such that mult $G\times G\to G$ and inverse $G\to G$ are smooth maps. G acts on $\mathcal{C}^\infty(G)$ by $(g\cdot\varphi)(h)=\varphi(hg)$, hence on the Lie algebra

$$\mathrm{Vect}(G) := \mathrm{Der}\,\mathcal{C}^\infty(G)$$

The Lie algebra of G is

$$\mathfrak{g}=\mathrm{Lie}\,G := \mathrm{Vect}(G)^G$$

There is a bijection between real fd Lie algebras and connected, simply connected Lie groups.

Every real Lie group G has a connected subgroup of the identity element. G° $\quad \mathrm{Lie}\, G = \mathrm{Lie}(G^\circ)$

Every connected Lie group G has a universal cover which is also a Lie group. $\tilde{G}$ $\quad \mathrm{Lie}\, G = \mathrm{Lie}(\tilde{G})$

Ex $\quad O(n)^\circ = SO(n)$

$\widetilde{SO(n)} = Spin(n)$

So-called spin group

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Spin(n) \to SO(n) \to 1$$

$Spin(n)$ is a double cover of $SO(n)$ (when $n > 2$)

$Spin(3) \cong SU(2)$

§2.8 Quivers.

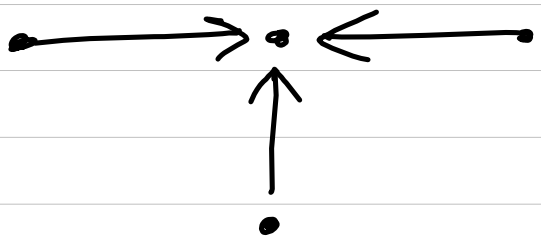
$Q = (Q_0, Q_1, s, t) = (I, E, s, t)$

$s, t : Q_1 \to Q_0$

source and target maps — arrows — vertices

$s(h) =: h'$, $t(h) =: h''$

Ex



$Q_0 = \{ a \quad b \quad c \quad d \}$

$Q_1 = \{ \alpha \quad \beta \quad \gamma \}$

$s(\alpha) = a \quad t(\alpha) = b \quad$ etc.

A rep of a quiver Q is $\{V_i\}_{i \in Q_0}$ and $\{X_h : V_{h'} \to V_{h''}\}_{h \in Q_1}$

Def The path alg $P_Q = kQ$ of a quiver Q is the vector space with basis all oriented paths in Q (including a trivial path p_i, $i \in I$ at each vertex) and mult def by $ab :=$ path b followed by path a if they connect and 0 otherwise

Remark $\sum_{i \in I} p_i = 1_{P_Q}$

A subrep of a rep (V_i, x_h) of a quiver is a rep (W_i, y_h) s.t. $W_i \subseteq V_i$ $\forall i \in I$ and $y_h = x_h|_{W_{i'}} : W_{h'} \to W_{h''}$ $\forall h \in E$

Def Direct sum, homomorphism of reps.

§3 General Results

§3.1 Subreps in ss reps

Assume $\bar{k}=k$

Def. A **semisimple** (or completely reducible) rep of an alg A is a direct sum of irreducible reps.

Ex Let V be an n-dim'l irrep of A. Then $Y := \mathrm{End}(V)$ with action of A by left mult. is a semisimple rep of A isomorphic to $nV := V \oplus \cdots \oplus V$. Indeed any basis $\{v_1, \ldots, v_n\}$ for V gives $\mathrm{End}(V) \to nV$ given by

$$x \longmapsto (xv_1, \ldots, xv_n)$$

This is an isomorphism of reps.

<u>Remark</u> If V is semisimple, there is a map

$$\phi : \bigoplus_{X \in \mathrm{Irr}A} \mathrm{Hom}_A(X, V) \otimes X \longrightarrow V$$

where X runs over a set of representatives of the set of isoclasses of irreps of A.

The map is given by

$$\phi(g \otimes x \longmapsto g(x))$$

$\forall g \in \mathrm{Hom}_A(X, V)$, $\forall x \in X$, $\forall X \in \mathrm{Irr}A$. Then ϕ is an isomorphism (use Schur's Lemma)

Prp 3.1.4

Let $\operatorname{Irr} A = \{V_1, \dots, V_m\}$

Let W be any subrep of $V := \bigoplus n_i V_i$ $(n_i \in \mathbb{Z}_{\geq 0})$

Then $W \cong \bigoplus r_i V_i$ where $r_i \leq n_i$ and the inclusion $W \hookrightarrow V$ is a direct sum of inj. homomorphisms $\phi_i : r_i V_i \to n_i V_i$ given by mult. from the right by some $r_i \times n_i$ matrix X_i with lin. indep. rows:

$$\phi_i(v_1, \dots, v_{r_i}) = (v_1, \dots, v_{r_i}) X_i$$

pf Ind. on $n = \sum_{i=1}^{m} n_i$

$n = 1$ clear

Spse $n > 1$. Assume $W \neq 0$.

Fix irr subrep $P \subset W$.

(exists by Pr 2.3!5)

By Schur's Lemma, $P \cong V_i$ for some i and $\phi|_P : P \to V$ factors through $n_i V_i$.

Identifying P with V_i, $\phi|_P$ is given by $v \mapsto (vq_1, \ldots, vq_{n_i})$ for some $q_\ell \in k \setminus \{0\}$.

Note $GL_i := GL_{n_i}(k)$ acts on $n_i V_i$ by $(v_1, \ldots, v_{n_i}) \mapsto (v_1, \ldots, v_{n_i})g$

and by id on $n_j V_j$ $j \neq i$

therefore on the set of subreps of V having property we seek, namely

$$X_i \longmapsto X_i g_i$$

$$X_j \longmapsto X_j \qquad j \neq i$$

Take $g_i \in G_{n_i}$ s.t.

$$(q_1, \ldots, q_{n_i}) g_i = (1, 0, \ldots, 0)$$

Then $W_{g_i} = V_i \oplus W'$ where

$W' \subset n_1 V_1 \oplus \cdots \oplus (n_i - 1) V_i \oplus \cdots \oplus n_m V_m$

is kernel of proj of W_{g_i} to first summand of V_i along the other summands. Done by induction.

∎

Rem Holds for general field & $\dim V = \infty$.

Cor 3.2.1 V fd irrep of A
Let $n>0$ and
let $v_1, \ldots, v_n \in V$ lin indep
Then $\forall w_1, \ldots, w_n \in V$
$\exists a \in A$ s.t. $av_i = w_i$

Pf Assume the contrary. Then the image of $A \to^n V$, $a \mapsto (av_1, \ldots, av_n)$ is a proper subrep so by Prp it corresponds to some rxn-matrix X, $r<n$. Taking $a=$,
$\exists u_1, \ldots, u_r \in V$ s.t.

$(u_1, \ldots, u_r)X = (v_1, \ldots, v_n)$

Let $(q_1, \ldots, q_n) \in k^n$ be nonzero s.t. $X\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$

(exists by $r < n$). Then

$$\sum q_i v_i = [u_1 \ldots u_r] X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$$

i.e. $\sum q_i v_i = 0 \Rightarrow$ contrad. ∎

Th. (Density Thm)

i) V f.dim irrep of A then

$\rho : A \to End(V)$ is surj

ii) $\bigoplus \rho_i : A \to \bigoplus End(V_i)$ surj

if $V = V_1 \oplus \ldots \oplus V_r$, $V_i \not\cong V_j$

Pf i) $B := \rho(A) \subset End(V)$.

WTS $B = End(V)$.

Let $c \in End(V)$

let v_i basis of V

let $w_i = c v_i$. By (or

$\exists a \in A$ s.t

$a v_i = w_i$

Then $a \mapsto c$ so $c \in B$.

∎

Lecture 9, Mon Feb 10

Example

A alg over $\bar{k}=k$.

$V = V_1 \oplus V_2$ direct sum of 2 irreps

$W \subseteq V$ an irreducible subrep.

What can we say about W?

Case 1 $V_1 \not\cong V_2$. Consider the projections

$$\pi_i : V_1 \oplus V_2 \longrightarrow V_i \quad , i=1,2$$
$$(v_1, v_2) \longmapsto v_i$$

π_i are intertwining operators.

Consider

$$\phi_i := \pi_i|_W : W \longrightarrow V_i \qquad i=1,2$$

If $\phi_1 \neq 0$, then it is an isomorphism by Schur's Lemma. So $W \cong V_1 \not\cong V_2$. This forces $\phi_2 = 0$. Symmetrically, if $\phi_2 \neq 0$ then $\phi_1 = 0$.

We conclude that

$$W = V_1 \oplus 0 \quad \text{or} \quad W = 0 \oplus V_2$$

(If both $\phi_1 = \phi_2 = 0$ then $W = 0$ contradicting W irr.)

CASE 2: $V_1 \cong V_2$

As in previous case, at least one of the projections π_i must be nonzero, hence $W \cong V_1 \cong V_2$.

$$End_A(V) = Hom_A(V_1 \oplus V_2, V_1 \oplus V_2)$$

$$\cong \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

where

$$H_{ij} = Hom_A(V_i, V_j)$$

By Schur's Lemma, $\dim_k H_{ij} = \underline{1}$.
Now $\exists \{e_{ij}\}$ basis for H_{ij} such that

$End_A(V) \cong M_2(\ )$ (exercise)

$\Rightarrow GL_2(k) = M_2(k)^\times \cong Aut_A(V)$ is acting on V.

Given any nonzero $w = (v_1, v_2) \in V$ we can find $g \in GL_2(k)$ "rotating" w to $(w_1, 0)$. Composing with π_2 gives non-injective (hence zero, by Schur's Lemma) intertwining operator $W \to 0 \oplus V_2$

This shows $\exists$ automorphism ϕ of V such that

$$\phi(W) = V_1 \oplus 0 \subset V_1 \oplus V_2 = V$$

Cor 3.2.1 A alg over $k = \bar{k}$
V fd irrep of A
$v_1, \dots, v_n \in V$ lin indep. ($\Rightarrow n \le \dim V$)
Then for any $w_1, \dots, w_n \in V$ $\exists a \in A$:
$a v_i = w_i \quad \forall i = 1, 2, \dots, n.$

Thm (Jordan Density Thm) A alg / $k = \bar{k}$
i) V fd irrep of A. Then
$\rho : A \to \mathrm{End}(V)$ is surjective
ii) $V = V_1 \oplus \cdots \oplus V_n$ V_i fd irreps,
$V_i \not\cong V_j$ $\forall i \neq j$. Then
$\rho = \oplus \rho_i : A \to \oplus \mathrm{End}(V_i)$ is surjective.

Pf i) $v_1, \dots, v_n$ basis for V,
$c \in \mathrm{End}(V)$. By cor $\exists a$ s.t.
$a v_i = c(v_i)$ $\forall i$. Thus $c = \rho(a)$
ii) $\oplus \mathrm{End}(V_i)$ is semisimple $\cong \oplus d_i V_i$
where $d_i = \dim V_i$. Thus by Prp 3.1.4
$\rho(A) = \oplus \rho_i(A) \overset{\text{ii)}}{=} \oplus \mathrm{End}(V_i)$ ∎

§3.3

$$A = \bigoplus_{i=1}^{r} Mat_{d_i}(k) = \begin{bmatrix} * & & \\ & * & \\ & & \ddots \end{bmatrix}$$

Thm The irreps of A are

$$V_1 = k^{d_1}, \ldots, V_r = k^{d_r}$$

and any fd rep of A is a direct sum of copies of V_i.

Def Dual rep $V^* \in {}_A\mathrm{or}\underline{\mathrm{Mod}}$.

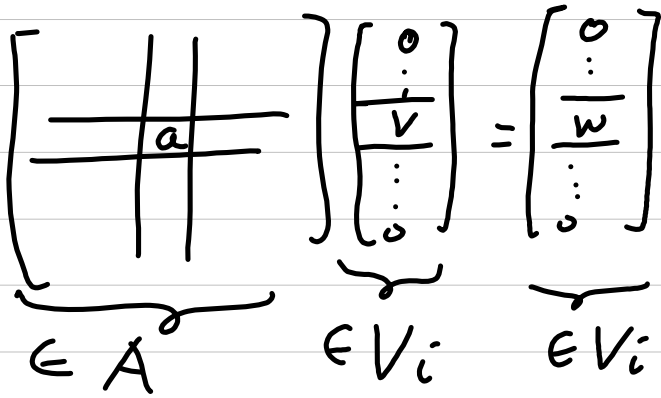
$$(a\xi)(v) = \xi(av)$$

Pf: Next time.

Lecture 10, Wed Feb 12, 2025

Proof of Thm (see previous page)

The representations V_i are irreducible: Let $v, w \in k^{d_i}$ be any nonzero vectors. Then there is $a \in Mat_{d_i}(k)$, $av = 0$. Therefore



Let X be an n-dim'l rep of A. Then X^* is a rep of A^{op}. But

$$A^{op} = \left(\oplus Mat_{d_i}(k)\right)^{op} =$$

$$= \oplus \left(Mat_{d_i}(k)^{op}\right)$$

Using the transpose $\rightarrow$

$$\cong \oplus Mat_{d_i}(k) = A$$

Fix a basis $\{y_i\}_{i=1}^n$ for X^*. Define

$$\phi: \underbrace{A \oplus \cdots \oplus A}_{n} \longrightarrow X^*$$

by

$$\phi(a_1, \ldots, a_n) = a_1 y_1 + \cdots + a_n y_n.$$

Since $k1_A \subseteq A$, ϕ is surjective.

Thus, the dual map

$$\phi^*: X \longrightarrow (A^n)^*$$

is injective.

Now $(A^n)^* \cong A^n$ as representations of A (using left mult, and transpose). Thus X is iso isomorphic to a subrep of A^n.

Since $A \cong \bigoplus_{i=1}^r d_i V_i$,

$$A^n \cong \bigoplus_{i=1}^r n d_i V_i$$

So $X \cong \bigoplus_{i=1}^r m V_i$ by Prop 3.1.4.

∎

§3.4 Composition Series.

Every fd rep V of an alg A has a <u>filtration</u>

$0 = V_0 \subset V_1 \subset \dots \subset V_\ell = V$

such that V_i/V_{i-1} is irreducible $\forall i = 1, \dots, \ell$.

Pf If V irr, done.

otherwise Let V_1 be an irr subrep. Let $W = V/V_1$

$\dim W < \dim V$ so by ind.

$\exists\ 0 = W_0 \subset W_1 \subset \dots \subset W_\ell = W$

Each W_i has the form $W_i = V_{i+1}/V_i$ for some subreps

V_{i+1} of V containing V_1.

By 3rd iso thm

$$V_{i+1}/V_i \cong W_i/W_{i-1} \quad \text{irr}$$

$\forall\ i = 1, \ldots, \ell$.

and $V_1/V_0 \cong V$. irr

∎

§ 3.5 Fin Dim'l Algs — Rad(A)

Def The **radical** of a finite-dim'l alg A is the set of $a \in A$ acting by zero on all irreps of A.

Prop 3.5.2 $\mathrm{Rad}(A)$ is a two-sided ideal.

Prop 3.5.3 A fin dim'l alg

i) I nilpotent ($I^n = 0, n >> 0$) 2-sided ideal of $A \Rightarrow I \subset \mathrm{Rad}\, A$

ii) $\mathrm{Rad}(A)$ is nilpotent

pf Let V irrep of A.
Let $v \in V$. Then $I_v \subset V$
is a subrep. If $I_v \neq 0$
then $Iv = V$ so there is
$x \in I$ st $xv = v$. Then
$x^n v = v$ so $x^n \neq 0$ contrad.

ii) $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$
composition series of
regular rep: A_i/A_{i-1} irr.
Let $x \in \operatorname{Rad} A$. Then
$x A_i \subset A_{i-1} \quad \forall i = 1, \ldots, n$
$\Rightarrow (\operatorname{Rad} A)^n = 0$ ∎

Th. A fd alg.

Then $\exists$ only fin many irreps (up to iso).

$V_1, \ldots, V_m$

These are f.d. and

$$A/\mathrm{Rad}A \cong \bigoplus_i \mathrm{End}\, V_i$$

Pf $A_r \subset V$ fd subsp. $\Rightarrow \dim V \leq \dim A$.

Spse

$V_1, \ldots, V_r$ noniso irreps

Then

$$\bigoplus \rho_i : A \to \bigoplus \mathrm{End}\, V_i$$

is surjective by Th 322.

So $r \leq \sum_i \dim \mathrm{End}\, V_i \leq \dim A$.

Let $\{V_1, \ldots, V_r\}$ all irreps.

By Th 322

$\bigoplus \rho_i : A \longrightarrow \bigoplus \mathrm{End}(V_i)$

is surjective.

kernel $= \mathrm{Rad}(A)$

by def. ∎

Cor $\sum (\dim V_i)^2 \leq \dim A$

Lecture 11

Ex. $A = k[x]/(x^n)$

$\exists!$ irrep $V = k$ with x acting by zero

So $\mathrm{Rad}(A) = (x)$.

Ex $A = \left\{ \begin{bmatrix} * & * & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}$ upper-triangular $n \times n$-matrices over k. Then $V = k^n$ has a composition series

$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$

given by

$V_i = \mathrm{Span}\{e_1, \ldots, e_i\}$

- Each V_i is a subrep of V
- Each V_i/V_{i-1} is 1-dim'l, hence irreducible.
- $\mathrm{Rad}(A) \supset \{$strictly upper-triangular matrices$\}$
-

$\uparrow \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ith row

Def A fin-dim'l alg A is **semisimple** if $\mathrm{Rad}(A) = 0$.

Prop 3.5.8. Let A be a fin-dim'l algebra. TFAE:

1) A is semisimple

2) $\dim A = \sum_{i=1}^{r} (\dim V_i)^2$ where $V_1, V_2, \ldots, V_r$ are the irreps of A.

3) $A \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$ for some $d_i \in \mathbb{Z}_{>0}$

4) Any finite-dimensional rep of A is completely reducible.

5) The regular rep. A is completely reducible.

Pf (1) $\Leftrightarrow$ (2): By Thm 3.5.4,

$$A/\mathrm{Rad}(A) \cong \bigoplus_i \mathrm{End}(V_i)$$

so $\dim A - \dim \mathrm{Rad}(A) = \sum_i (\dim V_i)^2$

(1) $\Rightarrow$ (3): By Th 3.5.4, if $\mathrm{Rad}(A)=0$ then

$$A \cong \bigoplus \mathrm{Mat}_{d_i}(k)$$

with $d_i = \dim V_i$.

(3) $\Rightarrow$ (1): By Th. 3.3.1, on irreps of $\bigoplus_i \mathrm{Mat}_{d_i}(k)$, no nonzero element acts by zero on all irreps, so $\mathrm{Rad}(A)=0$.

(3) $\Rightarrow$ (4): By Th. 3.3.1.

(4) $\Rightarrow$ (5): Trivial

(5) $\Rightarrow$ (3): Suppose $A = \bigoplus_i n_i V_i$.

By distributive law for $\mathrm{Hom}_A(-,-)$ w.r.t. $\oplus$:

$$\mathrm{End}_A(A) \cong \begin{bmatrix} H_{11} & \cdots & H_{1r} \\ \vdots & & \vdots \\ H_{r1} & \cdots & H_{rr} \end{bmatrix}$$

where $H_{ij} = \mathrm{Hom}_A(n_i V_i, n_j V_j)$.

By Schur's Lemma

$$H_{ij} = 0 \quad \text{for} \quad i \neq j$$

and

$$H_{ii} \approx Mat_{d_i}(k) \quad \text{for } i=j,\ d_i = \dim V_i$$

So

$$End_A(A) \cong \bigoplus Mat_{d_i}(k)$$

Since $End_A(A) \cong A^{op}$ (Exercise) we get

$$A \cong End_A(A)^{op} \cong \bigoplus Mat_{d_i}(k)^{op}$$

$$\cong \bigoplus Mat_{d_i}(k)$$

using transpose

∎

§ 3.6 Characters

$\rho_V : A \to End(V)$

Let A be an alg and (V, ρ_V) a fin-dim'l rep of A.

Def The character of V is the map $\chi_V : A \to k$ defined by $\chi_V(a) = Tr(\rho_V(a)) \quad \forall a \in A$.

Note Let

$$[A,A] = \text{Span}\{[a,b] \mid a,b \in A\}$$

Then $[A,A] \subset \ker \chi_V$ for any fd rep V so we can regard

(vector space, not alg)

$$\chi_V : \frac{A}{[A,A]} \longrightarrow k$$

Th i) Characters of distinct fd irreps are linearly indep

ii) If A is fd semisimple, these form a basis for $(A/[A,A])^*$.

Pf i) $V_1, \dots, V_r$ noniso irreps

Then $\oplus \rho_i : A \to \oplus \operatorname{End} V_i$

surj. by density Thm.

So $\chi_{V_1}, \dots, \chi_{V_r}$ are lin. indep.

ii) $[\mathrm{Mat}_d, \mathrm{Mat}_d] = \mathfrak{sl}_d$

Then

$$[A,A] = \bigoplus_{i=1}^{r} \mathfrak{sl}_{d_i}(k)$$

so $\dim (A/[A,A])^* = r$.

Lecture 12

§ 3.7 Jordan-Hölder Thm

Th: A any alg
V a fd rep of A
Let
$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$
and
$$0 = V_0' \subset V_1' \subset \cdots \subset V_m' = V$$
be two composition series for V
That is,
$$W_i = V_i/V_{i-1} \quad \text{and} \quad W_i' = V_i'/V_{i-1}'$$
are irreducible $\forall i$.
Then $m = n$, and $\exists$ permutation σ such that
$$W_i' \cong W_{\sigma(i)} \quad \forall i = 1, \ldots, n.$$

Proof We assume $\operatorname{char} k = 0$
(see book for general case)
$$\chi_V = \sum_{i=1}^{n} \chi_{W_i} = \sum_{i=1}^{m} \chi_{W_i'}$$
But characters of irreps are linearly independent. So the multiplicity of an irrep W among W_i and among W_i' is the same.

Def The "n" is the length of V.

§ 3.8 Krull–Schmidt Thm

Thm Any fd rep of A can be uniquely (up to isomorphisms and reordering summands) be decomposed into a direct sum of indecomposable reps.

Pf Existence is easy by induction. Suppose

$$V = V_1 \oplus \cdots \oplus V_m = V_1' \oplus \cdots \oplus V_n'.$$

Let $i_s : V_s \to V$ $\quad i_s' : V_s' \to V$

$p_s : V \to V_s$ $\quad p_s' : V \to V_s'$

be the inclusions and projections.

Let $\theta_s = p_1 i_s' p_s' i_1 : V_1 \to V_1$. Then $\sum_{s=1}^{n} \theta_s = \mathrm{Id}_{V_1}$ We will use Lemma:

Lemma 3.8.2

W fd indec rep of A
Then

i) Any homomorphism

$$\theta : W \longrightarrow W$$

is either an isomorphism or nilpotent.

ii) If $\theta_s : W \to W$ $s = 1, \ldots, n$ are nilpotent homomorphisms then so is $\theta := \theta_1 + \cdots + \theta_n$.

Proof i) Generalized eigenspaces of θ are subreps of W and W is their sum. Thus θ can only have one eigenvalue, say λ. If $\lambda = 0$, θ is nilpotent. If $\lambda \neq 0$, θ has $\neq 0$ determinant and thus is an isomorphism.

ii) By induction on n, we may assume $n=2$.

If $\theta = \theta_1 + \theta_2$ is not nilpotent, then by 1) it is an isomorphism. Then

$$Id_W = \theta^{-1}\theta_1 + \theta^{-1}\theta_2$$

$\theta^{-1}\theta_i$ are not isomorphisms ($\det = 0$) so they are nilpotent.

So $T = Id_W - \theta^{-1}\theta_1$ is an isomorphism (with inverse $1 + T + \cdots + T^{N-1}$, if $(\theta^{-1}\theta_1)^N = 0$) contradicting that $T = \theta^{-1}\theta_2$ is nilpotent.

∎

(Proof of Krull-Schmidt contd.)

By Lemma, θ_s is an isomorphism for some s.

Then $V_1' = \mathrm{Im}(p_1' i_1) \oplus \ker(p_1' i_1')$

so, since V_1' is indec,

$f := p_1' i_1 : V_1 \to V_1'$ and

$g := p_1 i_1' : V_1' \to V_1$ are

isomorphisms.

Let $B = \bigoplus_{j>1} V_j$ $\quad B' = \bigoplus_{j>1} V_j'$

Then

$$V = V_1 \oplus B = V_1 \oplus B' \quad (*)$$

Let

$h : B \to B'$ be $B \hookrightarrow V \twoheadrightarrow B'$ (along $(*)$)

Then h is an isomorphism: If $v \in \ker h$ then $v \in V_1'$ so $gv = 0 \Rightarrow v = 0$.
$\dim B = \dim B' \Rightarrow h$ iso.

Now use induction. ∎

Lecture 13 Wed Feb 19, 2025

§4 Representations of finite groups.

G finite group

k algebraically closed field but arbitrary characteristic

$k[G]$ or kG the group algebra of G

We will denote the basis for kG $\{g : g \in G\} = G$

Works well when G is <u>written multiplicatively</u>

Ex $G = S_2$ $kS_2 = \text{Span}\{(1), (12)\}$

$G = \mathbb{Z}_n \cong \langle t \rangle$, $t^n = 1$

$k\mathbb{Z}_n = \text{span}\{1, t, \ldots, t^{n-1}\}$

$\cong k[t]/(t^n - 1)$

Thm 4.1.1 (Maschke's Thm)

Suppose the characteristic of k does not divide $|G|$. Then

i) $k[G]$ is semisimple

ii) There is an algebra isomorphism

$$\psi : k[G] \longrightarrow \bigoplus_i End(V_i)$$

$$g \longmapsto g|_{V_i} := \rho_{V_i}(g)$$

where V_i are the distinct irreps of G. In particular, ψ is an isomorphism of representations, (G acts on both sides by left mult.)

Hence the regular rep. decomposes as

$$k[G] \cong \bigoplus_i (\dim V_i) V_i$$

and $$|G| = \sum_i (\dim V_i)^2$$

Proof By Prop 3.5.8, (i) $\Rightarrow$ (ii)

To prove (i), it suffices to show that if V is any fd rep of G and $W \subset V$ any subrep, then there is a subrep $W' \subset V$ such that $V = W \oplus W'$

First, choose any linear complement $\hat{W} \subset V$ of W in V.

Then $V = W \oplus \hat{W}$ as vector spaces but there is no reason $\hat{W}$ is a subrep.

Let $P : V \longrightarrow W$ be the projection onto W along $\hat{W}$. Then $P|_W = Id_W$, $P|_{\hat{W}} = 0$.

Define the symmetrized projection

$$\bar{P} = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1})$$

we can do this since $|G| \in k^\times$.

Now $\bar{P}|_W = Id_W$ and

$\bar{P}(V) \subseteq W$ so $\bar{P}^2 = \bar{P}$

Let $W' = \ker \bar{P}$

Then $V = W \oplus W'$.

We show $\bar{P}$ is an intertwining operator, hence W' is a subrep.

$\forall h \in G$:

$$\bar{P}\rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1}h) =$$

{Make the substitution $g = hl$.}

$$= \frac{1}{|G|} \sum_{l \in G} \rho(hl) P \rho(l^{-1}h^{-1}h)$$

$$= \rho(h) \bar{P}$$

∎

In fact, the converse holds:

Prop 4.1.2 If $k[G]$ is semisimple, then $\mathrm{char} k \nmid |G|$.

Pf Write $k[G] = \bigoplus_{i=1}^{r} \mathrm{End}\, V_i$

WLOG $V_1 = k$ is the trivial rep.

Then $k[G] = k \oplus \left(\bigoplus_{i=2}^{r} d_i V_i \right)$, $d_i = \dim V_i$

By Schur's Lemma

$$\mathrm{Hom}_{k[G]}(k, k[G]) = k\Lambda$$

$$\mathrm{Hom}_{k[G]}(k[G], k) = k\varepsilon$$

$\varepsilon : k[G] \to k$, $\Lambda : k \to k[G]$ nonzero maps of reps.

WLOG $\varepsilon(g) = 1 \ \forall g$ and $\Lambda(1) = \sum_{g \in G} g$

Then $\varepsilon \circ \underbrace{\Lambda(1)}_{\neq 0} = \varepsilon\left(\sum g\right) = \sum 1 = |G|$

$\Rightarrow |G| \neq 0$ in k i.e. $\mathrm{char} k \nmid |G|$. ∎

§4.2 Characters

Def A class function

$$f: G \to k$$

is a function that is constant on conjugacy classes.

Th 4.2.1 If $\mathrm{char}\, k \nmid |G|$, then the set of irreducible characters for G is a basis for the space of class functions.

Corollary The number of irreps of G is equal to the number of conjugacy classes of G.

Ex $G = S_3 \cong D_3$

$S_3 = \{(1), (12), (23), (13), (123), (132)\}$

C_1 C_2 C_3

There are three conjugacy classes.
So there are 3 irreps.
Always 1 trivial

$$|S_3| = \sum_{i=1}^{r} (\dim V_i)^2$$

$$\Rightarrow \quad 6 = 1^2 + d_2^2 + d_3^2$$

The only solution is

$$6 = 1^2 + 1^2 + 2^2$$

V_1 = trivial
V_2 = sign rep
$V_3 = k^2$, $\rho((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

belong to $k = \bar{k}$ char $k \neq 2, 3$

$$\rho((123)) = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Lecture 14 Fri Feb 21, 2025

$$F(G,k) = \{ f : G \to k \} \quad \text{(all fcns)}$$

$$F_c(G,k) = \{ f : G \to k \mid f(ghg^{-1}) = f(h) \ \forall g,h \in G \}$$

The characters $\chi_i = \chi_{V_i}$ of the irreducible reps $V_1, \dots, V_r$ of G are called **irreducible characters.**

Elements of $F_c(G,k)$ are the **class functions** of G.

Note: $\dim F_c(G,k) = \#$ conjugacy classes in G

Thm 4.2.1: When $\mathrm{char}\, k \nmid |G|$, $\{\chi_i\}_{i=1}^{r}$ is a basis for $F_c(G,k)$.

Pf $(A/[A,A]) \cong F_c(G,k)$

$(A/[A,A])$: space of all characters of A (Read in book)

(We previously showed that when A is semisimple, the irreducible characters are a basis for $(A/[A,A])^*$.)

Thus, for $A = k[G]$, when $\operatorname{char} k \nmid |G|$, the number of irr. characters of G is equal to the number of conjugacy classes in G.

Character Tables

Example: $G = S_3$

S_3	1	3	2
	(1)	(12)	(123)
$\chi_1 = \chi_{triv}$	1	1	1
$\chi_2 = \chi_{sgn}$	1	-1	1
χ_3	2	0	-1

← size of each conjugacy class

← representatives g_j of conjugacy classes

$\chi_i(g_j)$

$$\rho_3((1)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \chi_3((1)) = 2$$

$$\rho_3((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \chi_3((12)) = 0$$

$$\rho_3((123)) = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -1/2 & \\ & -1/2 \end{bmatrix}$$

$$\Rightarrow \chi_3((123)) = -1$$

General features:

G	1				
	e	g			
χ_{triv}	1	1	1	$\cdots$	1
χ_2	d_2	$*$	$*$	$\cdots$	$*$
χ_3	d_3	$*$	$\ddots$		$\vdots$
$\vdots$		$\vdots$		$\ddots$	$\vdots$
χ_r	d_r	$*$	$\cdots$	$\cdots$	$*$

where $d_i = \dim V_i$

Example $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

quaternion group

$ijk = i^2 = j^2 = k^2 = -1$ $\left(\begin{array}{c} ij = k = -ji \\ \Rightarrow jk = i = -kj \\ ki = j = -ik \end{array}\right)$

The conjugacy classes are

$\{1\}$ $\{-1\}$ $\{\pm i\}$ $\{\pm j\}$ $\{\pm k\}$

So there are 5 irreps one of which is the trivial.

Can they all be 1-dim'l?

No! Then $k[Q_8] \cong \bigoplus_{i=1}^{8} \mathrm{End}(V_i)$

$\cong k^8$ commutative!

$3^2 > 8$ so only way is

$8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$

Q_8	1	1	2	2	2
	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
	1	1	-1	1	-1
	1	1	-1	-1	1
χ_5	2				

By $Q_8 \longrightarrow Q_8/Z(Q_8) = Q_8/\{\pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

we get 4 1-dim'l irreps from those of $\mathbb{Z}_2 \times \mathbb{Z}_2$!

There all have: $\rho_\ell(-1) = 1$
$\ell = 1, 2, 3, 4$

Since $\rho_\ell(k) = \rho_\ell(i)\rho_\ell(j)$ and they are 1-dim'l,

$$\chi_\ell(k) = \chi_\ell(i)\chi_\ell(j) \quad \ell = 1, \dots, 4$$

So

$$\chi_\ell(i) = \pm 1, \ \chi_\ell(j) = \pm 1$$

§ 4.4. Duals & $\otimes$

Dual rep:

$$\rho_{V^*}(g) = \rho_V(g^*)^{-1} = \rho_V(g^{-1})^*$$

$$\Rightarrow \chi_{V^*}(g) = \chi_V(g^{-1})$$

Now

$$\chi_V(g) = \sum \lambda_i \quad , \quad \lambda_i \in \mathbb{C}, \ |\lambda_i| = 1$$

since $\lambda_i^{|G|} = 1$ by $\rho_V(g)^{|G|} = 1$

$$\chi_{V \otimes W} = \chi_V \chi_W$$

§4.5 Orthogonality of Characters.

Define a positive definite Hermitian inner product on $F_c(G, \mathbb{C})$ by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Th 4.5.1 $\forall$ reps V, W

$$(\chi_V, \chi_W) = \dim_G(V, W)$$

Thus

$$(\chi_V, \chi_W) = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}$$

pf. $(\chi_V, \chi_W) = \frac{1}{|G|} \sum \chi_V(g) \chi_{W^*}(g)$

$$= \frac{1}{|G|} \sum \chi_{V \otimes W^*}(g)$$

$$= \mathrm{Tr}_{V \otimes W^*}\left(\underbrace{\frac{1}{|G|} \sum g}_{p \in Z(\mathbb{C}G)} \right)$$

$P|_{triv} = 1$

$P|_{V} = 0$

V irr $\neq 1$

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Row-Orthogonality again:

1) If V, W fd vec spaces there is a natural isomorphism

$T: W \otimes (V^*) \longrightarrow \mathrm{Hom}(V, W)$

determined by

$W \times (V^*) \ni (w, \xi) \longmapsto \varphi_{w,\xi}$

where $\varphi_{w,\xi}(v) = \xi(v)\, w$.

2) If V, W are fd reps of a finite group G then

- V^* is a rep: $(g\xi)(v) = \xi(g^{-1}v)$

and

- $W \otimes (V^*)$ is a rep:
$g(w \otimes \xi) = gw \otimes g\xi$

- $\mathrm{Hom}(V, W)$ is a rep:
$(g\varphi)(v) = g(\varphi(g^{-1}v))$

and T is an isomorphism of representations.

3) If V is a rep of G, the subspace of <u>G-invariants</u> in V is

$$V^G = \{v \in V \mid gv = v \ \forall g \in G\}$$

<u>Note</u> $V^G = n_1 V_1$ if $V = \bigoplus n_i V_i$ with distinct irreps V_i and $V_1 = k$ the trivial rep of G.

4) When $\text{char } k \nmid |G|$,

$$P := \frac{1}{|G|} \sum_{g \in G} g \in G$$

is a special element in $Z(k[G])$ called the <u>symmetrizer</u>

If V is any fd rep of G then $\rho_V(P) = \frac{1}{|G|} \sum_g \rho_V(g)$

is equal to the projection of V onto $n_1 V_1 = V^G$ along $\bigoplus_{j>1} n_j V_j$.

5) If V, W fd reps of G, then

$$(\mathrm{Hom}(V,W))^G = \mathrm{Hom}_G(V,W)\,!$$

Indeed: $g\varphi = \varphi$

$\Leftrightarrow g(\varphi(g^{-1}v)) = \varphi(v) \quad \forall g \forall v$

$\Leftrightarrow g\varphi(v) = \varphi(gv) \quad \forall g \forall v.$

Now the proof:

$$(\chi_V, \chi_W) = \frac{1}{|G|} \sum_g \overline{\chi_V(g)} \chi_W(g)$$

I am officially redefining the inner product so conjugate is to the left. This is the correct way of doing it, as it aligns with left-slot-contravariance of the bifunctor $\mathrm{Hom}(-,-)$

(You can choose whatever convention you want)

(Note $(f,g) = \overline{(g,f)}$ either way.)

we continue:

$$(\chi_V, \chi_W) = \frac{1}{|G|}\sum_g \overline{\chi_V(g)}\,\chi_W(g)$$

$$= \frac{1}{|G|}\sum_g \chi_{V^*}(g)\,\chi_W(g)$$

$$= \frac{1}{|G|}\sum_g \chi_{V^*\otimes W}(g)$$

$$= \frac{1}{|G|}\sum_g \chi_{\mathrm{Hom}(V,W)}(g)$$

$$= \frac{1}{|G|}\sum_g \mathrm{Tr}\left(\rho_{\mathrm{Hom}(V,W)}(g)\right)$$

$$= \mathrm{Tr}\,\rho_{\mathrm{Hom}(V,W)}(P) =$$

P ↑ the G-symmetrizer

$$= \dim\left(\mathrm{Hom}(V,W)^G\right)$$

$$= \dim \mathrm{Hom}_G(V,W)$$

∎

Column-Orthogonality

Th 4.5.4.

Let $Z_g = \{ z \in G \mid zg = gz \}$ be the centralizer of g in G.

Then $\forall g, h \in G$:

$$\sum_{V \in \mathrm{Irr}\, G} \overline{\chi_V(g)}\, \chi_V(h) = \begin{cases} |Z_g| \\ \\ 0 \end{cases}$$

$$= \begin{cases} |Z_g| & \text{if } g \text{ is conjugate to } h \\ \\ 0 & \text{otherwise.} \end{cases}$$

(In particular,

$$\frac{1}{|G|} \sum_{V \in \mathrm{Irr}\, G} |Cl(g)|\, \overline{\chi_V(g)}\, \chi_V(h) = 1$$

$Cl(g) = \{ xgx^{-1} \mid x \in G \}$

since $|Cl(g)| = |G : Z_g| = |G| / |Z_g|$.)

Pf $\sum_V \overline{\chi_V(g)}\chi_V(h) =$

$= \sum_V \mathrm{Tr}(\rho_{V^*}(g))\,\mathrm{Tr}(\rho_V(h))$

$= \sum_V \mathrm{Tr}(\rho_{V^*\otimes V}(g\otimes h))$

$k[G]\otimes k[G]$

$= \mathrm{Tr}\left(\rho_{\bigoplus_V \mathrm{End}(V)}(g\otimes h)\right)$

$= \mathrm{Tr}_{\bigoplus_V \mathrm{End}(V)}\left(\varphi \longmapsto \rho(g)\circ\varphi\circ\rho(h^{-1})\right)$

$= \mathrm{Tr}_{k[G]}\left(x \longmapsto gxh^{-1}\right)$

$= \#\{x\in G : gxh^{-1}=x\}$

$= \#\{x\in G : x^{-1}gx = h\}$

$= \begin{cases} |Z_g| & g \text{ conj. to } h \\ 0 & \end{cases}$

∎

Ex D_4 dihedral group of order 8

$$D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

Conjugacy classes

$$\underbrace{\{1\}, \{r^2\}}_{Z(D_4)}, \{r, r^3\}, \{sr, sr^3\}, \{sr^2, s\}$$

So there are 5 irreps

$$\Rightarrow 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$$

D_4	1	1	2	2	2	
	1	r^2	r	sr	s	
χ_1	1	1	1	1	1	$D_4 \to S_4 \xrightarrow{\text{sgn}} \{\pm 1\}$
χ_2	1	1	-1	1	-1	
χ_3	1	1	1	-1	-1	$D_4/Z(D_4)$
χ_4	1	1	-1	-1	1	$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$
χ_5	2	-2	0	0	0	

Lecture 16 Feb 26, 2025

§ 4.6 Assume $k = \mathbb{C}$.

Def A fd rep V of G is unitarizable if there is a G-invariant positive definite Hermitian form $(\cdot, \cdot)$ on V.

(G-invariant means $g.(\cdot,\cdot) = (\cdot,\cdot)$ $\forall g$

i.e. $(\rho_V(g)u, \rho_V(g)v) = (u, v)$

$\forall g \in G \quad \forall u, v \in V$

Thm G finite, V fd rep of G. Then V is unitarizable.

Def $(V, (\cdot,\cdot))$ is a unitary rep.

§ 4.7 Orthogonality of Matrix elements.

Let $(V, (\cdot,\cdot))$ be a unitary fd rep of G, $|G| < \infty$
Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V.

Def The matrix elements $t_{ij}^V : G \to \mathbb{C}$ are defined by

$$t_{ij}^V(g) = (v_j, \rho_V(g) v_i) \quad , g \in G$$

On the space $F(G, \mathbb{C})$ of all functions from G to $\mathbb{C}$ we define

$$(f, g) = \sum_{x \in G} \overline{f(x)}\, g(x)$$

Prop 4.7.1 If V, W are irreps of G,

$$(t_{ij}^V, t_{ke}^W) = \frac{\delta_{V \cong W}}{\dim V} \delta_{ik} \delta_{jl}$$

Thus, the matrix elements

$$\{ t^V_{ij} \mid V \in \mathrm{Irr}\, G,\ i,j = 1,\dots,\dim V \}$$

is an orthonormal basis for $F(G,\mathbb{C})$.

Multiplicities and Tensor Products

If $V \cong m_1 V_1 \oplus \cdots \oplus m_r V_r$

with V_i irreps of G then

$$\chi_V = \sum_{i=1}^{r} m_i \chi_i \qquad (\chi_i = \chi_{V_i})$$

By orthogonality,

$$(\chi_j, \chi_V) = \sum_i m_i \underbrace{(\chi_j, \chi_i)}_{\delta_{ij}}$$

$$= m_j$$

so

$$\boxed{m_j = (\chi_j, \chi_V)}$$

Special Case:

$$V = V_k \otimes V_\ell$$

Then

$$m_{j'} = (\chi_j, \chi_k \chi_\ell).$$

Ex $G = Q_8$ (or D_4)

	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Question Let $V_5 = \mathbb{C}^2$ be the 2-dim'l irrep.
How does $V_5 \otimes V_5$ decompose?

$$(\chi_5, \chi_5 \chi_5) = \frac{1}{8}\left(\chi_5(1)^3 + \chi_5(-1)^3\right)$$

$$= \frac{1}{8}(8-8) = 0$$

$i = 1,2,3,4$

$$(\chi_i, \chi_5\chi_5) = \frac{1}{8}(1\cdot(2)^2 + 1\cdot(-1)^2) = 1$$

Conclusion: $m_1 = m_2 = m_3 = m_4 = 1$, $m_5 = 0$

$$V_5 \otimes V_5 \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

Ex character table for A_4

Note:

- $(123) \sim (132)$ in S_4 but not in A_4
- $N = \{(1), (12)(34), (13)(24), (14)(23)\} \trianglelefteq A_4$ and $A_4/N = \langle (123)N \rangle \cong \mathbb{Z}_3$
  $\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \; g$
- There are 4 conjugacy classes with representatives: (1), $(12)(34)$, (123), (132)
- $|A_4| = \frac{4!}{2} = 12$
- We get 3 one-dimensional irreps from $\mathbb{Z}_3$

$$A_4 \xrightarrow{\pi} A_4/N \xrightarrow{\overline{\rho_i}} \mathbb{C}^\times$$

$$\langle g \rangle, g \longmapsto \varepsilon^{i-1}, \; i = 1,2,3$$

$$\varepsilon = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$$

$$\rho_i := \overline{\rho}_i \circ \pi \qquad i = 1, 2, 3$$

$$12 = 1^2 + 1^2 + 1^2 + c^2 \Rightarrow c = 3$$

So A_4 has a 3-dim'l irrep

So far we know:

A_4	1	3	4	4
	(1)	$(12)(34)$	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ε	ε^2
χ_3	1	1	ε^2	ε
χ_4	3			

By column orthogonality, we now get

$$\chi_4((12)(34)) = -1$$

$$\chi_4((ijk1)) = 0$$

So last row is

χ_4	3	-1	0	0

Lecture 17 Fri Feb 28, 2025

$|G| < \infty$ V fd, $k = \mathbb{C}$

§5.1 Frobenius-Schur Indicator

V cplx if $V \not\cong V^*$

real if $\exists$ nondeg G-inv. symm bil. form on V

quaternionic if ... skew-sym

. . . . .

Def The Frobenius-Schur indicator of a rep V of G is

$$FS(V) = \begin{cases} 0 & V \text{ cplx} \\ 1 & V \text{ real} \\ -1 & V \text{ quat.} \end{cases}$$

Th (F-S) The number of involutions (= elements of order ≤ 2) of G is

$$\sum_{V \in \mathrm{Irr}\, G} (\dim V) \cdot FS(V)$$

pf If $A: V \to V$ has eigenvalues

$\lambda_1, \ldots, \lambda_n$ then

$$\mathrm{Tr}|_{S^2 V}(A \otimes A) = \sum_{i \le j} \lambda_i \lambda_j$$

$$\mathrm{Tr}|_{\Lambda^2 V}(A \otimes A) = \sum_{i < j} \lambda_i \lambda_j$$

Thus

$$\mathrm{Tr}|_{S^2 V}(A \otimes A) - \mathrm{Tr}_{\Lambda^2 V}(A \otimes A) =$$

$$= \sum_i \lambda_i^2 = \mathrm{Tr}(A^2)$$

Thus for $g \in G$

$$\chi_V(g^2) = \chi_{S^2 V}(g) - \chi_{\Lambda^2 V}(g)$$

Let $P = \frac{1}{|G|} \sum_{g \in G} g$ be the symmetrizer. Then

$$|G|^{-1}\chi_V(\Sigma g^2) = \chi_{S^2V}(P) - \chi_{\Lambda^2 V}(P) =$$

$$= \dim (S^2V)^G - \dim (\Lambda^2 V)^G =$$

$$= \begin{cases} 1 & V \text{ real} \\ -1 & V \text{ quat} \\ 0 & V \text{ cplx} \end{cases} \qquad (*)$$

The number of involutions in G equals (by column orthogonality)

$$\frac{1}{|G|} \sum_{V \in \operatorname{Irr} G} \dim V \, \chi_V(\Sigma g^2) =$$

$$\overset{(*)}{=} \sum_{\text{real } V} \dim V - \sum_{\text{quat } V} \dim V$$

$$\sum_V \overbrace{\chi_V(1)}^{\dim V} \chi_V(g^2) = \begin{cases} z_{g^2} & g^2 \sim 1 \\ 0 & g^2 \not\sim 1 \end{cases}$$

$$= \begin{cases} |G| & g^2 = 1 \\ 0 & g^2 \neq 1 \end{cases}$$

§5.2 Algebraic Numbers and Algebraic Integers.

Def $z \in \mathbb{C}$ is an **algebraic number** (resp. **algebraic integer**) if it's a root of a monic pol with rational (resp. integer) coefficients

Equivalently, if z is an eigenvalue of a determinant with rational (resp. integer) entries.

Indeed, $\Leftarrow$ is clear while for $\Rightarrow$ use companion matrix.

Notation

$\overline{\mathbb{Q}}$ = algebraic numbers

$\mathbb{A}$ = — " — integers

Prop i) $\mathbb{A}$ is a subring of $\mathbb{C}$ containing $\mathbb{Z}$

ii) $\overline{\mathbb{Q}}$ is a subfield of $\mathbb{C}$ containing $\mathbb{Q}$.

Pf α eig. val of A $n \times n$

β eig. val of B $m \times m$

Then

$\alpha \pm \beta$ eig. val. of $A \otimes I_m \pm I_n \otimes B$

and

$\alpha \cdot \beta$ eig. val. of $A \otimes B$.

(eigenvector is $v \otimes w$ if $Av = \alpha v$ and $Bw = \beta w$).

If $\alpha \neq 0$ then α^{-1} root of $x^d p_\alpha(1/x)$.

$\uparrow$

$p_\alpha(\alpha) = 0$

∎

Prop $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$

Pf skip

Lem If $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}}$ then all alg. conjugates of $\sum \alpha_i$ are $\sum \alpha_i'$ where α_i' conj. to α_i.

§ 5.3 Frobenius divisibility

G finite, V irrep / $\mathbb{C}$

Then

$\dim V$ divides $|G|$

Lecture 18 Mon, Mar 3, 2025

Induction and Restriction

Forgetting mult. in a group H gives a set $S(H)$.
Conversely starting with a set X, we can construct the free group $F(X)$

These two constructions are clearly not inverse, but thats ok.

$$\mathrm{Grp} \underset{\text{Free}}{\overset{\text{Forget}}{\rightleftarrows}} \mathrm{Set}$$

Instead we have a universal property: Any set map into a group extends to the free group. More precisely:

$$\mathrm{Hom}_{\mathrm{Set}}(X, S(H)) \cong \mathrm{Hom}_{\mathrm{Grp}}(F(X), H)$$

We say F is left adjoint to S and write $F \dashv S$.

(More properties are required in the full definition.)

For a finite group G, let $\mathrm{Rep}\, G$ be the category of all finite-dimensional representations of G.

If $\varphi: H \to G$ is a group homomorphism (e.g. inclusion of a subgroup or a quotient map) and $\rho: G \to GL(V)$ is a rep of G, then $\rho \circ \varphi$ is a rep of H, denoted

$$V\big|^{\varphi}_{VH} \quad \text{or} \quad \mathrm{Res}^{G}_{H} V$$

If $T: V_1 \to V_2$ is a map of reps of G, the same map

$$T: \mathrm{Res}^G_H V_1 \to \mathrm{Res}^G_H V_2$$

is a map of reps of H.

So we have a functor

$$\underline{\mathrm{Rep}}\, G \xrightarrow{\mathrm{Res}^G_H} \underline{\mathrm{Rep}}\, H$$

What is the "freest" way of constructing a rep of G from a rep of H?

To answer this we pass to modules over group algebras

$$\underline{\mathrm{Rep}}\, G \cong {}_{kG}\underline{\mathrm{Mod}}$$

Let A, B be algebras with a fixed algebra map $\varphi: B \to A$. Using f, we make A into an (A, B)-bimodule via

$$a \triangleright x \triangleleft b = axf(b) \quad \forall a, x \in A, b \in B.$$

For any (left) B-module V we define

$$V\uparrow_{\varphi}^{A} = \mathrm{Ind}_{B}^{A} V = A \underset{B}{\otimes} V$$

$\underset{B}{\otimes}$ means $a \otimes bv = af(b) \otimes v$

When $A = \bigoplus_{i=1}^{d} a_i B$, we have

$$A \underset{B}{\otimes} V \cong \left(\bigoplus_{i=1}^{d} ka_i\right) \underset{k}{\otimes} V$$

Thm

$$\mathrm{Hom}_B(V, \mathrm{Res}_B^A W) \cong \mathrm{Hom}_A(\mathrm{Ind}_B^A V, W)$$

Specializing to group algebras:

If $H \leq G$ then $\varphi: H \hookrightarrow G$ gives

$$\mathrm{Hom}_H(V, \mathrm{Res}_H^G W) \cong \mathrm{Hom}_G(\mathrm{Ind}_H^G V, W)$$

Take dimensions

$$(\chi_V, \chi_{W\downarrow_H}) = (\chi_{V\uparrow G}, \chi_W)$$

for any $V \in \mathrm{Rep}\, H$, $W \in \mathrm{Rep}\, G$.

This is known as Frobenius Reciprocity.

Example

Let's induce up the sign representation of S_{n-1} to S_n and see what we get.

We need to consider kS_n as a right kS_{n-1}-module.

$$[S_n : S_{n-1}] = \frac{n!}{(n-1)!} = n \quad \text{so}$$

there are n cosets $g_i S_{n-1}$ making up S_n. Explicitly, if $\sigma \in S_n$ with $\sigma(n) = i$ then $(i\,n)\circ\sigma \in S_{n-1}$. Thus,

$$S_n = \bigsqcup_{i=1}^{n} (i\,n) S_{n-1}$$

Therefore $\mathbb{C}S_n$ is free as a right $\mathbb{C}S_{n-1}$-module with basis $\{(1), (12), (13), \ldots, (1n)\}$.

So $$\mathrm{Ind}_{S_{n-1}}^{S_n} \mathbb{C}_{sgn} \cong \left(\bigoplus_{i=1}^{n} \mathbb{C}(in)\right) \otimes_{\mathbb{C}} \mathbb{C}_{sgn}$$

If $\sigma \in S_n$ then

$$\sigma\left((in) \otimes 1_{sgn}\right)$$

$$= \left(\sigma \circ (in) \otimes 1_{sgn}\right)$$

we must write

$$\sigma \circ (in) = (jn) \circ \tilde{\sigma}$$

for some $\tilde{\sigma} \in S_{n-1}$, $j \in \{1, \ldots, n\}$

Then

$$\sigma\left((in) \otimes 1_{sgn}\right) =$$

$$= (jn) \circ \tilde{\sigma} \otimes 1_{sgn}$$

$$= (jn) \otimes (\tilde{\sigma} \cdot 1_{sgn})$$

$$= (jn) \otimes sgn\,\tilde{\sigma}\, 1_{sgn}$$

$$= (sgn\,\tilde{\sigma}) \cdot (jn) \otimes 1_{sgn}$$

Lorentz §4.7 Lecture 26

Schur Weyl Duality

$A \circlearrowright W$

$A_W = \rho_W(A) \subset End(W)$

$A'_W = End_A(W)$

$A''_W = \{ X \in End_A(W) \mid Xa = aX \ \forall a \in A'_W \}$

$=: BiEnd_A(W)$

Note $Irr\, A_W \subset Irr\, A$

by pullback $A \twoheadrightarrow A_W$.

$D(S) := End_A(S) \quad (= A'_S)$

Schur division alg

$\mathrm{Hom}_A(S, W)$ is an $(A'_W, D(S)\text{-bimod})$

$\Rightarrow \mathrm{Hom}_A(S, W) \otimes_{D(S)} S$

is a rep of A'_W

in fact of $A'_W \otimes A$.

Notation: $\mathrm{Hom}_A(S, W) \boxtimes_{D(S)} S$

W is also $A'_W \otimes A$ module

Th. 4.29 A semisimple

$W \in \mathrm{Rep}\, A$ finitely gen.

a) A_W, A'_W ss

b) $S \leftrightarrow \mathrm{Hom}_A(S, W)$ is a bij $\mathrm{Irr}\, A_W \leftrightarrow \mathrm{Irr}\, A'_W$

c) $W \cong \bigoplus_{S \in \mathrm{Irr}\, A_W} \mathrm{Hom}_A(S, W) \otimes S$ $D(S)$

in $\mathrm{Rep}(A'_W \otimes A)$ and terms are pairwise non isomorphic irreps

d) $A_W = A''_W$.

§ 4.7.2 V fd vsp.

S_n vs $GL(V)$

$k GL(V) \xrightarrow{\sigma} End_{S_n}(V^{\otimes n}) \twoheadrightarrow E$

$k S_n \xrightarrow{\pi} End_{GL(V)}(V^{\otimes n}) \twoheadrightarrow E$

$E = End_k(V^{\otimes n})$

Th. $\lambda \in \mathcal{P}_n$ $d = \dim V$

a) $S^\lambda V \in \mathrm{Rep}\, GL(V)$ is irr if $\ell(\lambda) \le d$

b) $\lambda \neq \mu \Rightarrow S^\lambda \not\cong S^\mu$

c) $V^{\otimes n} \cong \bigoplus S^\lambda V \boxtimes V^\lambda$

$S^W V \cong Hom_{S_n}(W, V^{\otimes n})$

Lecture 27

Quiver Representations.

(Etingof Ch. 6)

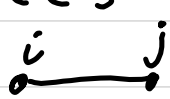
Def A **Dynkin diagram** is a graph Γ with

- finite vertex and edge sets
- multiple edges allowed 
- no self-loops 

such that the following matrix is positive definite:

$$A_\Gamma = 2I_N - R_\Gamma$$ (called the **Cartan matrix**)

where $N = \#$ vertices of Γ, and

$(R_\Gamma)_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$  = **adjacency matrix**

Example

A_N: 

Then $A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$

Sylvester's Criterion:

A necessary and sufficient condition for an $N \times N$-matrix A to be pos. def. is that

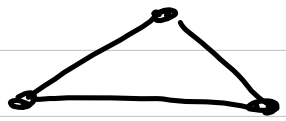
$$\det A^{(i)} > 0 \quad \forall i = 1, \ldots, N$$

where $A^{(i)} = (A_{kl})_{k,l=1}^{i}$ is the $i \times i$ upper left submatrix of A.

Note $(A_N)^{(i)} = A_i$ so suffixes here to show $\det A_i \neq 0 \ \forall i > 0$

(left as exercise) $\Rightarrow A_N$ is a Dynkin diagram.

Example

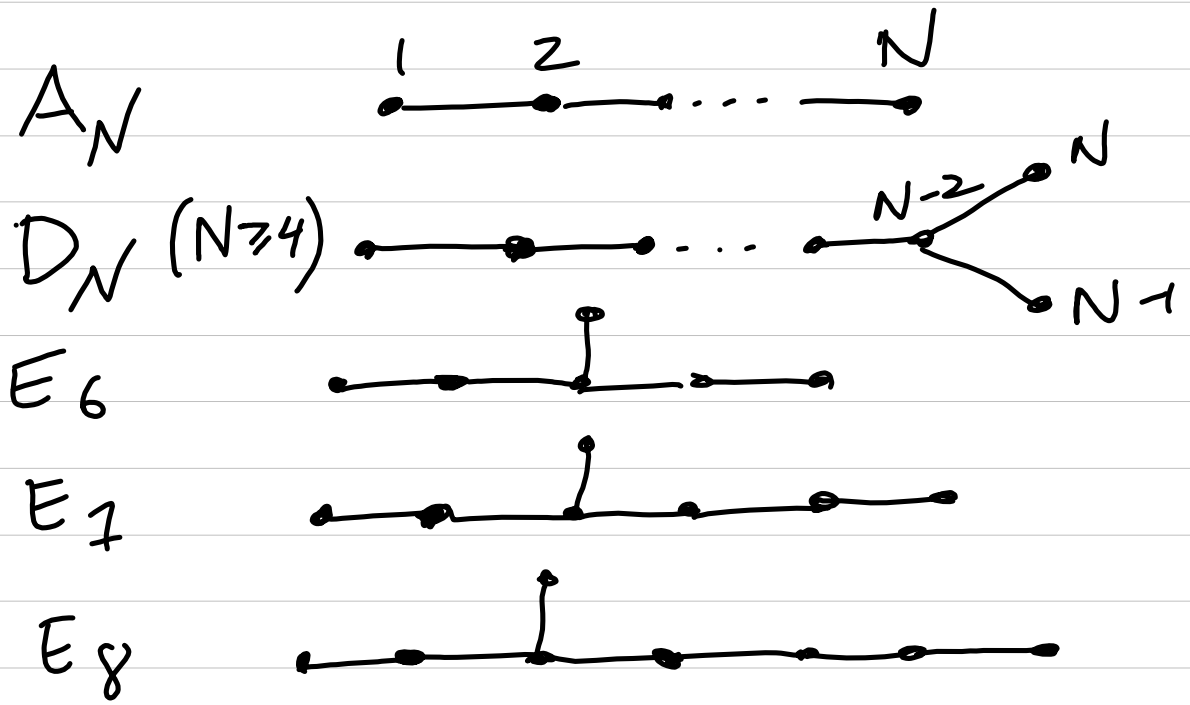
Γ: 

$$A_\Gamma = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \ker A_\Gamma$,

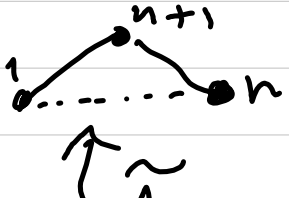
$\det A_\Gamma = 0$ so Γ is Not a Dynkin diagram.

Theorem The following are all possible Dynkin diagrams:



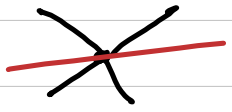
Pf (Sketch) a), b) Compute determinants and show $A_N D_N E_{6,7,8}$ are indeed Dynkin diagrams

c) If Γ Dynkin then



it contains no cycles

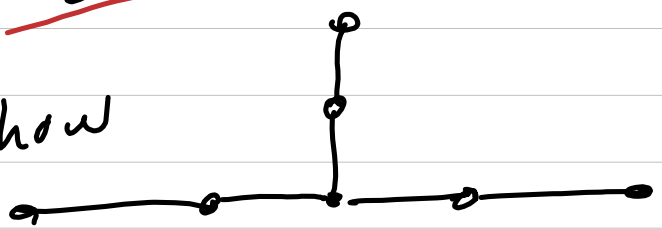
(otherwise $\ker A_\Gamma \neq 0$)

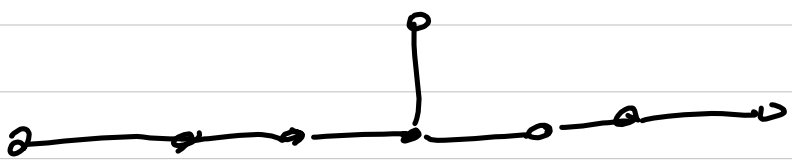
d) No vertex can be incident to $\geq$ four edges 

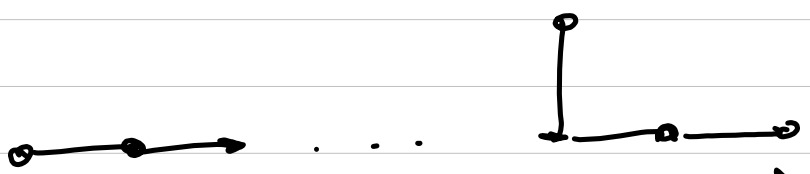
Also, at most one vertex can have 3 edges incident



e) Show







(9 vertices total)

have $\det(A_\Gamma) = 0$

f) Deduce classification.

Def The (simply laced) affine Dynkin diagrams are those with A_n pos. semidef.

$\tilde{A}_N$, $\tilde{D}_N$, $\tilde{E}_{6,7,8}$ are the extra ones

---

Def A quiver Q is of finite type if the set of isoclasses of indecomp. reps is finite.

Gabriel's Thm

Q is of finite type iff forgetting directions we get a Dynkin diagram

Lecture 28

Recall A rep of a quiver Q assigns a vector space to each vertex and a linear map to each arrow.

We have natural notions of $\oplus$ and indecomposable reps.

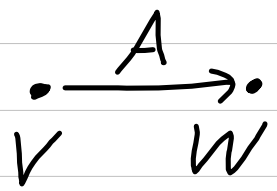
Ex $\overline{Q} = A_1$ •

The only indec. rep is

•
1

meaning •$^{k^1}$

Ex A_2 

A rep is 

, $A: V \to W$ linear

To decompose it, let V' be a (linear) complement to $\ker A$ in V: $V = \ker A \oplus V'$

Similarly, let W' be a complement to $\operatorname{Im} A$ in W.

Then



These are multiples of



respectively. (cf. Gaussian elimination)

Ex There are three quivers with underlying undirected graph A_3:

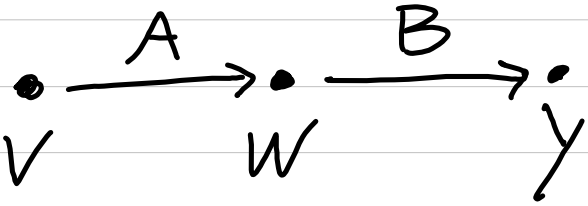
(1) 

(2) 

(3) 

Let's consider (1) first:

A rep looks like



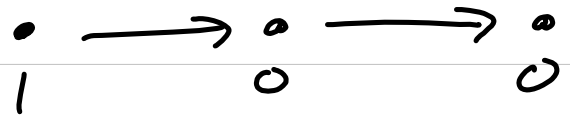
As before, we can split away



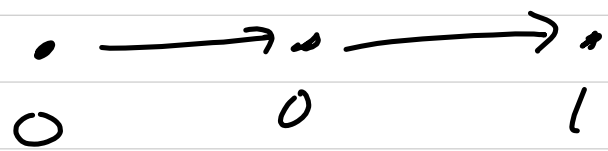
and



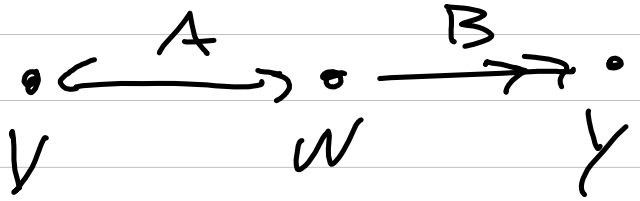
These are multiples of



and



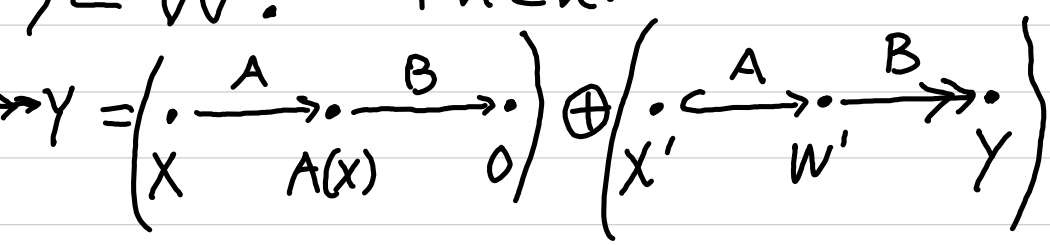
respectively. Thus WLOG



Let $X = \ker(B \circ A)$

Choose $X' \subset V$ st. $V = X \oplus X'$

Then choose a complement W' to $A(X)$ in W such that $A(X') \subset W'$. Then:

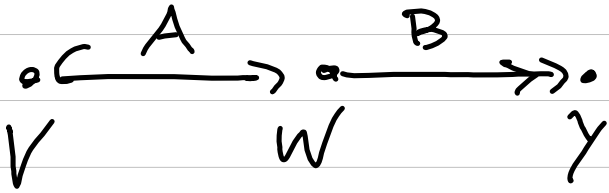


The first is a multiple of



The second has $\ker(B \circ A) = 0$

So WLOG we may assume this. Starting over:



and $\ker(B \circ A) = 0$.

Let $X = \mathrm{Im}(B \circ A)$ and $X' \subset Y$ a complement of X in Y.

{ Let's abuse notation and write $X' = Y \ominus X$ for "X' is a linear complement to X in Y" because we do this a lot. (This is a ternary relation) }

Further, let $W' = B^{-1}(X')$

Then $W = A(V) \oplus W'$ and

we get

$$\bullet_V \xrightarrow{A} \bullet_W \xrightarrow{B} \bullet_X = \left(\bullet_V \xrightarrow[\sim]{A} \bullet_{A(V)} \xrightarrow[\sim]{B} \bullet_X \right) \oplus$$

$(\ker B \circ A = 0)$

$$\oplus \left(\bullet_0 \longrightarrow \bullet_{W'} \xrightarrow{B} \bullet_{X'} \right)$$

The first summand is a multiple of $\bullet_1 \xrightarrow{\sim} \bullet_1 \xrightarrow{\sim} \bullet_1$

By splitting away $\ker(B)$ the second summand is a multiple of $\bullet_0 \longrightarrow \bullet_1 \xrightarrow{\sim} \bullet_1$

plus a multiple of $\bullet_0 \longrightarrow \bullet_1 \longrightarrow \bullet_0$

Thus, $\bullet \longrightarrow \bullet \longrightarrow \bullet$ has <u>six</u> indecomposable reps:

(2) $\bullet \longrightarrow \bullet \longleftarrow \bullet$

Similarly to (1), we can split away multiples of

$$\underset{1}{\bullet} \longrightarrow \underset{0}{\bullet} \longleftarrow \underset{0}{\bullet} \qquad \underset{0}{\bullet} \longrightarrow \underset{0}{\bullet} \longleftarrow \underset{1}{\bullet}$$

to get $\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet}$

Choose $W' = W \ominus (V \cap Y)$, let $V' = W' \cap V$, $Y' = W' \cap Y$. We get

$$\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet} =$$

$$= \underset{V'}{\bullet} \hookrightarrow \underset{W'}{\bullet} \hookleftarrow \underset{Y'}{\bullet} \oplus \underset{V\cap Y}{\bullet} \xrightarrow{\sim} \underset{V\cap Y}{\bullet} \xleftarrow{\sim} \underset{V\cap Y}{\bullet}$$

Second is $\oplus$ $\xrightarrow{\sim}$ $\xleftarrow{\sim}$;

first has $V' \cap Y' = 0$

So WLOG

$$V \hookrightarrow W \hookleftarrow Y$$

with $V \cap Y = 0$ in W.

Then $W' := W \ominus (V \oplus Y)$

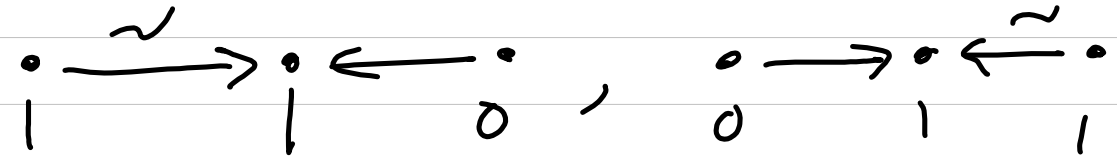
We can split off $\dim W'$ copies of $0 \to 1 \leftarrow 0$

and thus WLOG $W = V \oplus Y$

Now $V \hookrightarrow V \oplus Y \hookleftarrow Y$ =

$$= \left(V \xrightarrow{\sim} V \leftarrow 0\right) \oplus \left(0 \to Y \xleftarrow{\sim} Y\right)$$

These are multiples of



So again we get $\underline{\text{six}}$ indec. reps.

$$B(x,y) = x^T A_\Gamma y$$

A_Γ ← cartan matrix of Γ

Lemma Let Γ be a Dynkin diagram.

1) B pos. def.

2) $B(x,x) \in 2\mathbb{Z}_{>0}$ $\forall x \in \mathbb{Z}^n$

Pf 1) by def of Dynkin

2) $B(x,x) = 2\sum x_i^2 + \sum_{i \neq j} x_i a_{ij} x_j$

$$= 2\sum x_i^2 + 2\sum_{i<j} x_i a_{ij} x_j$$

∎

Def A root in $\mathbb{Z}^n$ (wrt B) is a shortest length vector

(i.e. $B(x,x) = 2$)

(thus # roots $< \infty$)

Def The vectors

$$\alpha_i = (0, \ldots, \overset{i\text{:th pos.}}{1}, \ldots, 0) \in \mathbb{Z}^n$$

are the simple roots

(Note $(\alpha_i, \alpha_i) = \alpha_i^T B \alpha_i = 2$

so they are roots)

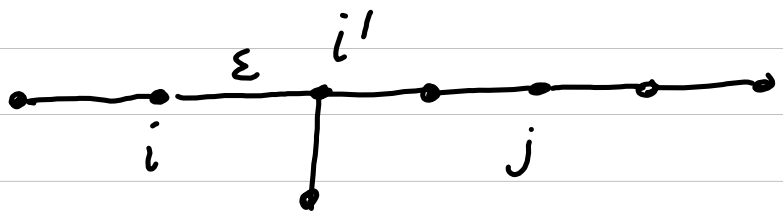
Lem 6.4.6 Let Γ be a Dynkin diagram with n vertices, and $\alpha \in \mathbb{Z}^n$ a root (wrt. $B(x,y) = x^T A_\Gamma y$)

Write $\alpha = \sum_{i=1}^{n} k_i \alpha_i$. Then

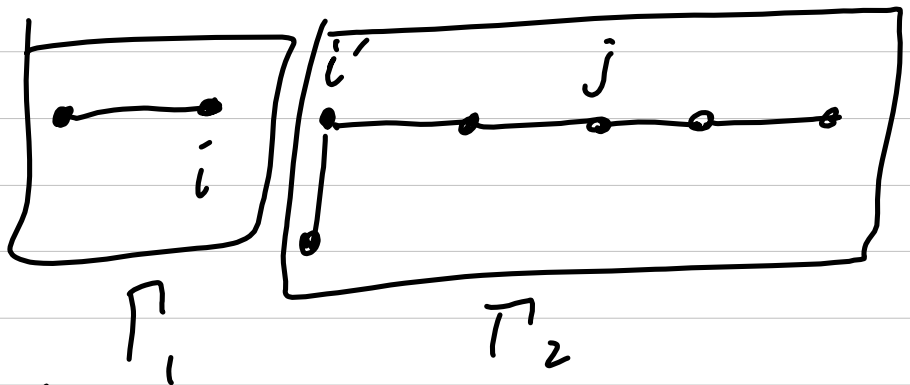
either $k_i \geq 0 \ \forall i$,

OR $k_i \leq 0 \ \forall i$.

Proof Assume the contrary i.e. that $k_i > 0$, $k_j < 0$ for some $i \neq j$. WLOG $k_s = 0$ for all s between i and j in Γ.



Let ε be the edge connecting i with the next vertex towards j, and i' be the other vertex of ε. Let Γ_1, Γ_2 be the graphs obtained by removing ε.



Define

$$\beta = \sum_{m \in \Gamma_1} k_m \alpha_m \qquad \gamma = \sum_{m \in \Gamma_2} k_m \alpha_m$$

(i.e. m is a vertex of Γ_1)

Then $\alpha = \beta + \gamma$.

Since $k_i > 0$, $k_j < 0$ we know $\beta \neq 0$, $\gamma \neq 0$.

Thus $B(\beta, \beta) \geq 2$ and $B(\gamma, \gamma) \geq 2$.

Also, from $B(\alpha_i, \alpha_{i'})$

$$B(\beta, \gamma) = -k_i k_{i'}$$

because Γ_1, Γ_2 are only connected by $\mathcal{E}$.

But $k_i > 0$ and $k_{i'} \leq 0$

($k_{i'} < 0$ if $i' = j$, otherwise $k_{i'} = 0$)

This yields

$$B(\alpha, \alpha) = B(\beta+\gamma, \beta+\gamma) =$$

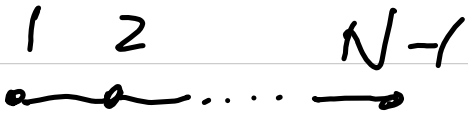
$$= \underbrace{B(\beta, \beta)}_{\geq 2} + 2\underbrace{B(\beta, \gamma)}_{\geq 0} + \underbrace{B(\gamma, \gamma)}_{\geq 2} \geq 4$$

contradicting that α is a root.



Def A root $\alpha = \sum k_i \alpha_i$ is positive if $k_i \geq 0 \ \forall i$ and negative if $k_i \leq 0 \ \forall i$.

Example

1) $\Gamma = A_{N-1}$ 

There is a way to understand where B comes from: $\mathbb{Z}^{N-1} \cong L \hookrightarrow \mathbb{Z}^N$

by $L = \{(x_1, \dots, x_N) \mid \sum x_i = 0\}$

$\mathbb{Z}^{N-1} \ni \alpha_1 \longmapsto (1, -1, 0, \dots, 0) = \varepsilon_1 - \varepsilon_2$

$\vdots$

$\mathbb{Z}^{N-1} \ni \alpha_{N-1} \longmapsto (0, \dots, 0, 1, -1) = \varepsilon_{N-1} - \varepsilon_N$

where $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^N$

This is an isometric embedding, when we equip $\mathbb{Z}^N$ with $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

$$(\varepsilon_i - \varepsilon_{i+1}, \varepsilon_j - \varepsilon_{j+1}) = \begin{cases} 2 & |i-j|=0 \\ -1 & |i-j|=1 \\ 0 & |i-j|>1 \end{cases}$$

$$= (\alpha_i, \alpha_j)$$

In L, $(x,x) = 2$ iff

$$x = \pm(0, \dots, \underset{i}{1}, \dots, \underset{j}{-1}, \dots, 0)$$

which corresponds to $\pm$

$$\beta_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

Thus there are $\frac{N(N-1)}{2}$ positive roots.

2) # positive roots :

D_N	$N(N-1)$
E_6	36
E_7	63
E_8	120

Def Let $\alpha \in \mathbb{Z}^n$ be a positive root (for a Dynkin diagram Γ with n vertices)
The reflection S_α is defined by

$$S_\alpha(v) = v - B(v, \alpha)\alpha$$

$\forall v \in \mathbb{Z}^n$.

Denote s_{α_i} by s_i and call them simple reflections

Note $s_\alpha(\alpha) = -\alpha$

and $s_\alpha(v) = v$ if $B(v, \alpha) = 0$

Thus s_α is a reflection in $\alpha^\perp := \{v \in \mathbb{R}^n \mid B(v, \alpha) = 0\}$

and preserves B.

Def
The Weyl group of Γ is the subgroup W of the orthogonal group $O(\mathbb{R}^n, B)$ generated by the s_i:

$$W = \langle s_1, \ldots, s_n \rangle$$

§6.5 Gabriel (again)

Def Let Q be a quiver with vertices $1, 2, \ldots, n$

Let $V = (V_1, V_2, \ldots, V_n)$ be a rep of Q. The dimension vector of V is

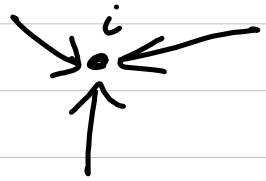
$$d(V) = (\dim V_1, \dim V_2, \ldots, \dim V_n)$$

Note In Problem 6.1.5 it is shown that if the underlying graph of a quiver is NOT a Dynkin diagram, then there are ∞ many indec. reps. of Q.

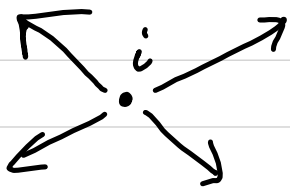
Th. 6.5.2 (Detailed Gabriel's Th.)
Let Q be a quiver of type A_N, D_N, E_6, E_7, or E_8. Let V be an indecomposable rep. of Q. Then $d(V)$ is a positive root (w.r.t. B_Γ). Conversely, for any root α there is exactly one (up to iso) indecomposable rep. of Q with dimension vector α.

[E, § 6.6] Reflection Functors

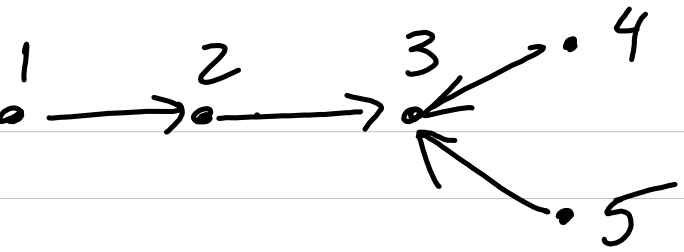
Def Q any quiver. Call a vertex $i \in Q$ a **sink** if all arrows connected to i point towards i:



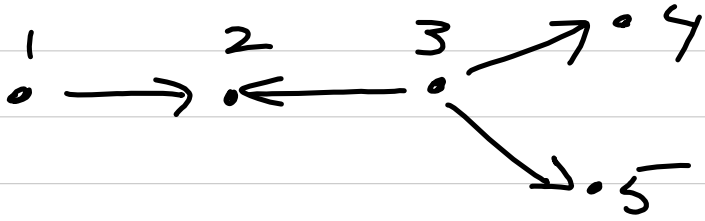
Analogously we define **source**.



Def Let Q be a quiver and $i \in Q$ a sink (resp. source) we denote by $\overline{Q}_i$ the quiver obtained from Q by reversing all arrows into (resp. out of) i.

Ex Q: 

Here 1 is a source, 3 is a sink, and 2 is neither.

$\overline{Q}_3$: 

In $\overline{Q}_3$, 2 is now a sink.

Def Let Q be a quiver and $i \in Q$ a sink. The reflection functor $F_i^+ : \mathrm{Rep}\, Q \to \mathrm{Rep}\, \overline{Q}_i$ is defined as follows: If $V = (V_k)_k$ is a representation of Q, put

$$(F_i^+ V)_k = \begin{cases} V_k & , k \neq i \\ \ker\left[\varphi : \bigoplus_{j \to i} V_j \to V_i\right], & k = i \end{cases}$$

Also, the maps are the same except those into V_i:

$\alpha : V_j \to V_i$ is replaced by

$\tilde{\alpha} : (F_i^+ V)_i \to V_j$ given by the composition

$$(F_i^+ V)_i = \ker \varphi \xrightarrow{\text{incl.}} \bigoplus_{j \to i} V_j \xrightarrow{\text{proj.}} V_j$$

(This defines the functor F_i^+ on objects. We leave it as exercise to define it on morphisms.)

Def Let Q be a quiver and $i \in Q$ a source. The reflection functor

$$F_i^- : \mathrm{Rep}\, Q \longrightarrow \mathrm{Rep}\, \overline{Q}_i$$

is defined by

$$(F_i^- V)_k = \begin{cases} V_k & k \neq i \\ \mathrm{Coker}\, \psi & k = i \end{cases}$$

where $\psi : V_i \longrightarrow \bigoplus_{i \to j} V_j$

and $\mathrm{Coker}\, \psi = \left(\bigoplus_{i \to j} V_j\right) / \mathrm{Im}\, \psi$.

Linear maps are the same except all $V_i \to V_k$ (maps out of i) are replaced by

$$V_k \hookrightarrow \bigoplus_{i \to j} V_j \twoheadrightarrow \mathrm{Coker}\, \psi.$$

Prop 6.5.6 Let Q quiver and V be an indec. rep. of Q.

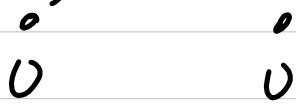
1) If $i \in Q$ is a sink, then either $\dim V_i = 1$, $\dim V_j = 0$ for $j \neq i$; or $\varphi : \bigoplus_{j \to i} V_j \to V_i$ is surj

2) If $i \in Q$ is a source then either ....

or $\varphi : V_i \to \bigoplus_{i \to j} V_j$ is injective

Proof 1) Choose a complement W of $\operatorname{Im}\varphi$. Then

$$V = \left(U \xrightarrow{\varphi} W \xleftarrow{0} 0 \right) \oplus V'$$



Either first or second summand has to be zero.

2) Similar. ∎

Prop 6.6.6 Q quiver, $V \in \mathrm{Rep}\, Q$

1) If $\varphi : \bigoplus_{j \to i} V_j \to V$ surj (i sink)

then $F_i^- F_i^+ V = V$

2) If i source & φ inj

then $F_i^+ F_i^- V = V$

Pf We only show $F_i^- F_i^+ V = V$

Suffices $(F_i^- F_i^+ V)_i = V_i$

Suppose φ is surj., let

$K = \ker \varphi$. Then $(F_i^+ V)_i = K$.

Also, let $\psi : K \to \bigoplus_{j \to i} V_j$

Then $(F_i^- F_i^+ V)_i = (F_i^- K)_i = K'$

where $K' = \left(\bigoplus_{j \to i} V_j\right) / \mathrm{im}\, \psi$

But $\operatorname{Im}\varphi = K$.

So $K' = \left(\bigoplus_{j\to i} V_j\right) / \ker\left(\varphi : \bigoplus_{j\to i} V_j \to V_i\right)$

$= \operatorname{Im}\left(\varphi : \bigoplus_{j\to i} V_j \to V_i\right)$

Since φ is surj, $K' = V_i$ ∎

Prop 6.6.7 Q quiver V indec rep of Q. Then $F_i^{\pm} V$ (when defined) are either 0 or indecomposable.

Pf Spse i sink, consider $F_i^+ V$

By Prop 6.6.5 $\varphi : \bigoplus_{j\to i} V_j \to V_i$ is either surj or $\dim V_i = 1$ $\dim V_j = 0$ $j \neq i$.

In last case $F_i^+ V = 0$.

In first case (φ surj), say

$F_i^+ V = X \oplus Y \qquad X, Y \neq 0$

But $F_i^+ V$ inj at i

so X, Y inj at i too

So $F_i^+ F_i^- X = X$ same w. Y

In part. both $\neq 0$.

So $V = F_i^- F_i^+ V = F_i^- X \oplus F_i^- Y$

Contrad. ∎

Lecture 31

(Proof of Prop 6.6.8)

Lecture 32

Let Q be a quiver whose underlying graph is a Dynkin diagram.

Corollary 6.8.4 For every positive root α there is an indecomposable rep V of Q with $d(V) = \alpha$.

Proof Consider the sequence of roots

$\alpha, s_n(\alpha), s_{n-1}s_n(\alpha), \ldots$

Let β be the first root in the sequence preceeding a negative root. Then $s_i(\beta)$ is a negative root, for some i.

But $s_i(\beta) = \beta - B(\beta, \alpha_i)\alpha_i$ only changes the i:th coordinate of β.

So $\beta = \alpha_i$ and thus

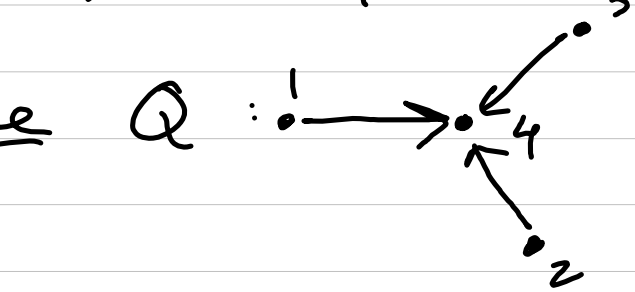
$(s_q s_{q+1} \cdots s_{n-1} s_n)(\alpha) = \alpha_i$

$(\Leftrightarrow \alpha = (s_n s_{n-1} \cdots s_q)(\alpha_i))$

for some q. Let V_{α_i} denote the (irreducible) rep of the quiver $(\cdots(Q_n)_{n-1}\cdots)_q$ with dimension vector α_i. Define

$$V = F_n^- F_{n-1}^- \cdots F_q^- V_{\alpha_i} \in \mathrm{Rep}\, Q$$

Then V is indecomposable and $d(V) = \alpha$. ∎

Example Q: 

$\Gamma = D_4$

$$F_4^- F_1^- F_2^- F_3^- V_{\alpha_4} = F_4^- V_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$$

$$= V_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4} \in \mathrm{Rep}\, Q$$

Roots of D_4?

$$A_\Gamma = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 2 & & & -1 \\ & 2 & & -1 \\ & & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix} \end{array}$$

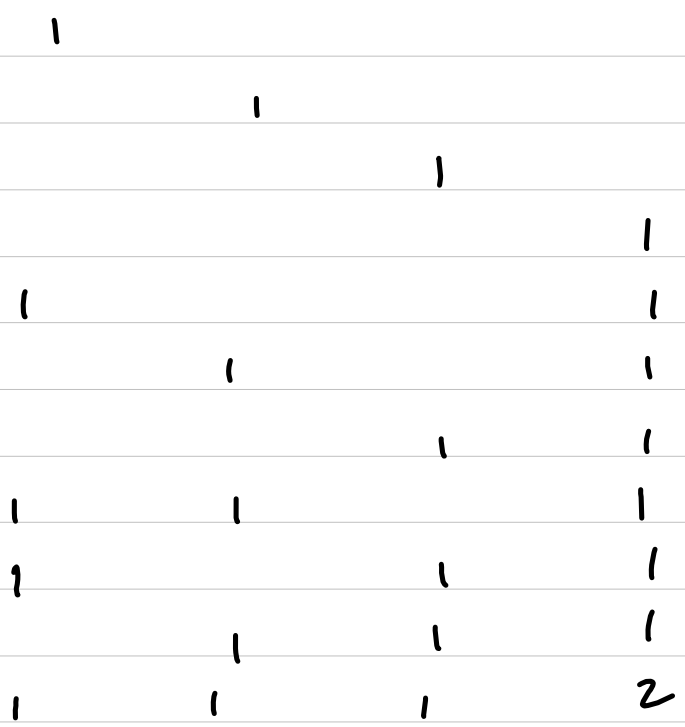
$$\beta = \sum k_i \alpha_i, \quad k_i \in \mathbb{Z}_{\geq 0}$$

$$B_\Gamma(\beta, \beta) = \sum_{ij} k_i k_j a_{ij}$$

$$= \sum_i 2k_i^2 - 2(k_1 + k_2 + k_3)k_4$$

When is this 2?

k_1 k_2 k_3 k_4

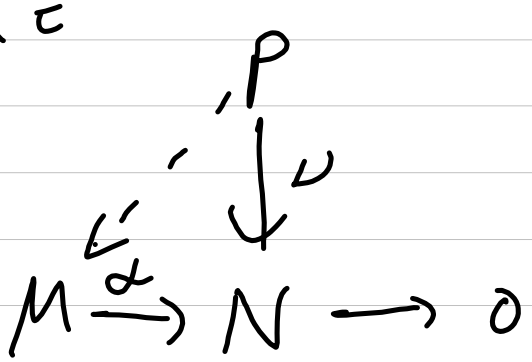


Lecture 35

§ 8 Homological Alg

A alg P left A-mod
TFAE

1)



2) Any surj $M \to P$ splits

3) $\exists\, Q$ $P \oplus Q = A^{\oplus I}$

4) $\mathrm{Hom}_A(P, \cdot)$ exact

Proof

1) $\Rightarrow$ 2) $N = P$

2) $\Rightarrow$ 3) M free

3) $\Rightarrow$ 4) $\mathrm{Hom}_A(A^{\oplus I}, \cdot)$ exact

4) $\Rightarrow$ 1) $K = \ker \alpha$ Apply $\mathrm{Hom}(P, \cdot)$ to $0 \to K \to M \to N \to 0$

Def P projective.

Dual notion: injective

P proj $\Leftrightarrow$ P^* inj right mod (fd)

Def Projective object of ab. cat: $\mathrm{Hom}(P, -)$ is exact.

§8.2 Tor & Ext (A = algebra) Lecture 36

Def A **projective resolution** of an A-module M is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

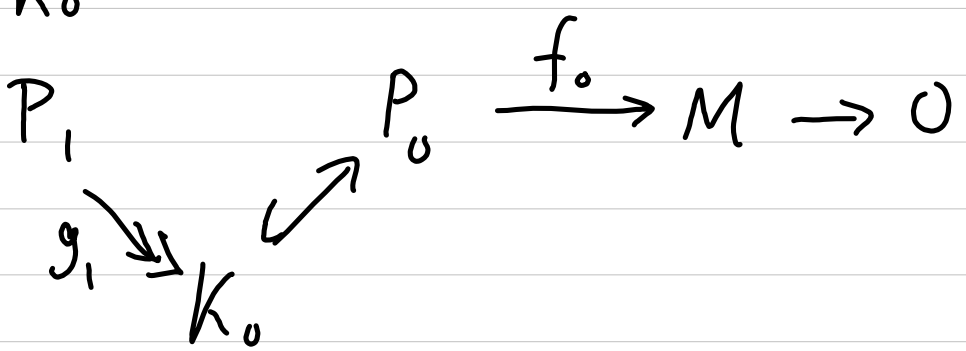
such that all P_i, $i \geq 0$, are projective A-modules.

Exercise 8.2.2: Any A-module has a projective resolution.

Proof Let P_0 be a free A-module that surjects onto M: $P_0 \xrightarrow{f_0} M \to 0$

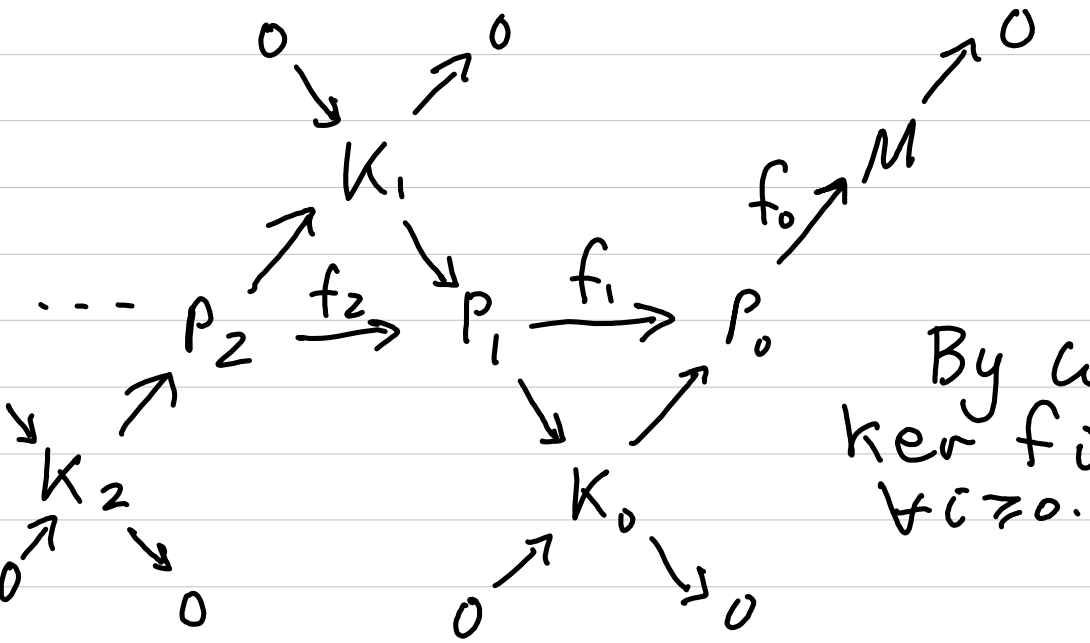
Let $K_0 = \ker f_0$.

Now let P_1 be a free A-module that surjects onto K_0: $g_1 : P_1 \twoheadrightarrow K_0$



Define $f_1 : P_1 \to P_0$ as the composition $P_1 \twoheadrightarrow K_0 \hookrightarrow P_0$

Let $K_1 = \ker f_1$ and repeat:



By construction $\ker f_i = \operatorname{im} f_{i+1}$ $\forall i \geq 0$.

∎

Def Let $P_\bullet$ be a projective resolution of a right A-module M. Let N be a left A-module.

For $i \geq 0$ we define

$\mathrm{Tor}_i^A(M,N)$ to be the i-th cohomology of the complex

$$\longrightarrow P_2 \underset{A}{\otimes} N \longrightarrow P_1 \underset{A}{\otimes} N \longrightarrow P_0 \underset{A}{\otimes} N \longrightarrow 0$$

induced by the complex $\cdots \to P_1 \to P_0 \to 0$.

Ex $A = \mathbb{Z}$ $\quad M = \mathbb{Z}/2\mathbb{Z}$, N any $\mathbb{Z}$-mod

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$\begin{array}{ccccccc} & & P_1 \otimes N & & P_0 \otimes N & & \\ & & \| & & \| & & \\ 0 & \to & \mathbb{Z} \underset{\mathbb{Z}}{\otimes} N & \xrightarrow{\cdot 2} & \mathbb{Z} \underset{\mathbb{Z}}{\otimes} N & \to & 0 \end{array}$$

$2\mathbb{Z}$ sits between the two rows.

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, N) = \frac{\ker(\cdot 2)}{0} = \{n \in N \mid 2n = 0\}$$

Ex $A = k$ field.

$$0 \to \underset{\| \; P_0}{M} \xrightarrow{\mathrm{id}} M \to 0$$

is a proj. resolution of M.

So $\mathrm{Tor}_i^A(M,N) = 0 \quad \forall N \; \forall i \geq 0$

Ex A any alg, $M = P$ projective

Then $0 \to P_0 = P \xrightarrow{\mathrm{id}} P \to 0$

is a proj. resolution of P

so $\mathrm{Tor}_i^A(P,N) = 0 \quad \forall N \; \forall i \geq 0.$

Def Let $P_\bullet$ be a projective resolution of a left A-module M. Let N be a left A-module. Then $\mathrm{Ext}^i_A(M,N)$ is defined as the i:th cohomology of the complex

$$0 \to \mathrm{Hom}_A(P_0, N) \xrightarrow{d_0} \mathrm{Hom}_A(P_1, N) \to \cdots$$

obtained by applying $\mathrm{Hom}_A(-, N)$ to the complex

$$\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

$= f_1^*$

Note $\mathrm{Ext}^0_A(M,N) = \dfrac{\ker d_0}{\mathrm{im}\, 0} \cong \ker d_0$

$= \{\varphi \in \mathrm{Hom}(P_0, N) \mid f_1^*(\varphi) = 0\} = \{\varphi \mid \varphi \circ f_1 = 0\}$

$\cong \mathrm{Hom}(P_0/\mathrm{im}\, f_1, N) \cong \mathrm{Hom}(P_0/\ker f_0, N) \cong \mathrm{Hom}_A(M,N)$.

Similarly $\mathrm{Tor}_0^A(M,N) = M \underset{A}{\otimes} N$.

Problem 8.2.5: Tor and Ext are independent of the choice of resolution.

FACT:

SES $\Rightarrow$ LES on cohomology:

Given a three term exact sequence (short exact seq, S.E.S.)

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

of left A-modules, we get a "long" exact seq (L.E.S.)

$$0 \to \mathrm{Hom}(M,N_1) \to \mathrm{Hom}(M,N_2) \to \mathrm{Hom}(M,N_3) \to$$
$$\xrightarrow{\Delta_1} \mathrm{Ext}^1(M,N_1) \to \mathrm{Ext}^1(M,N_2) \to \mathrm{Ext}^1(M,N_3) \to$$
$$\xrightarrow{\Delta_2} \mathrm{Ext}^2(M,N_1) \to \cdots$$

Δ_i are <u>connecting maps</u>

Abelian category case:

If every object M is a quotient of a projective object we say $\mathcal{C}$ has enough projectives

If so, we can define Ext^i and Tor_i (as proj. resolutions exist.)

Lecture 37 Fri Apr 25, 2025

§9. F.D. Algs

$I^n = 0, n >> 0.$

A = ring, $I \subset A$ nilpotent (2-sided) ideal

Prp 9.11. Let $e_0 \in A/I$ be an idempotent, i.e. $(e_0)^2 = e_0$.

Then $\exists e \in A$ such that

$e^2 = e$ e is idempotent

$e + I = e_0$ (e is a lift of e_0)

Such e is unique up to conjugation by any element of $1 + I$.

proof First assume $I^2 = 0$.

$\Rightarrow$ I bimod over A/I

Let $e_* \in A$ be any lift of e_0. Then $a := e_*^2 - e_* \in I$

and $e_0 a = a e_0$

Try $e = e_* + b$, $b \in I$

Then $e_0 b + b e_0 - b = a$

$b := (2e_0 - 1)a$ Then

$e_0 b + b e_0 - b = \cdots = a$ OK

Spse $e' = e + c$

e' idemp $\Rightarrow$ $ec + ce - c = 0$

$\Leftrightarrow$ $ece = 0$ & $(1-e)c(1-e) = 0$

so $c = ec(1-c) + (1-e)ce$

$= [e\,[e\,c]]$.

Thus $e' = (1 + [c, e])\, e\, (1 + [c, e])^{-1}$

General case

Ind on k, $I^k = 0$

$e_{m-1} \in A/I^m$

$\Rightarrow e_m \in A/I^{m+1} = A$

Def Compl syst of orth idemp in A

$e_1, \ldots, e_n \in B$

$e_i e_j = \delta_{ij}$ $\sum e_i = 1$

Cor $e_{01}, \ldots, e_{0n}$ compl sys orth idem in A/I

Then $\exists\, e_1 \ldots e_n$ in A lifting e_{0i}

Lecture 38 (Mon Apr 28)

Etingof §9.2

Suppose A is finite-dimensional. Every finitely generated A-module P is a quotient of A^n for some finite $n > 0$:

$$\pi : A^n \twoheadrightarrow P$$

Therefore $\dim P < \infty$ and P decomposes into a finite direct sum of indecomposable modules:

$$P = P_1 \oplus \cdots \oplus P_m$$

If furthermore P is projective then P, hence each P_i, is a direct summand of a free A-module. Therefore each P_i is also projective.

$\Rightarrow$ To understand finitely generated projective A-modules, it suffices to understand the indecomposable

projective modules.

Let A be a finite-dimensional algebra. Let $\{M_1, \ldots, M_n\} = \mathrm{Irr}\, A$.

Thm 9.2.1

i) For each $i = 1, 2, \ldots, n$ there is a unique indecomposable finitely generated projective A-module P_i such that

$$\mathrm{Hom}_A(P_i, M_j) = \delta_{ij} \qquad (\ast)$$

ii) $A \cong \bigoplus (\dim M_i) P_i$ as left A-modules

iii) Any indecomposable finitely generated projective A-module is isomorphic to P_i for some i.

Proof Recall $A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^{n} \mathrm{End}_A(M_i)$ and $\mathrm{Rad}(A)$ is a nilpotent ideal of A.

Pick a basis for M_i and let $e^{\circ}_{ij} = E^{i}_{jj}$ $\quad i=1,\dots,n$, $j=1,\dots,\dim M_i$ be the rank 1 projectors onto the basis vectors.
Then e°_{ij} are orthogonal idempotents of $A/\mathrm{Rad}(A)$.

By Cor. 9.3 we can lift them to orthogonal idempotents e_{ij} in A. Define

$$P_{ij} = Ae_{ij}$$

Then $A = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\dim M_i} P_{ij}$ which shows the P_{ij} are projective.

Also, $\mathrm{Hom}_A(P_{ij}, M_k) = e_{ij}M_k =$

$$= \begin{cases} 0 & i \neq k \\ 1\text{-dim'l}, & i = k \end{cases}$$

By uniqueness of e_{ij} up to conjugation under $A^{\times}$, $P_{ij} \cong P_{ik}$ for any j,k. Put $P_i = P_{i1}$.

We claim P_i are indecomp.
If $P_i = Q_1 \oplus Q_2$ then $\exists \ell \in \{1,2\}$ such that for all $j \in \{1,2,..,n\}$:

$$\mathrm{Hom}_A(Q_\ell, M_j) = 0$$

(otherwise $\dim \mathrm{Hom}_A(P_i, M_i) > 1$)

$\Rightarrow$ either $Q_1 = 0$ or $Q_2 = 0$.

Lastly any indecomposable projective module has to occur in the decomposition of A.

■

Def P_i is the projective cover of M_i.

Prop 9.2.3

Let N be any f.d. A-module. Then

$$\dim \mathrm{Hom}_A(P_i, N) = [N : M_i]$$

where RHS means the multiplicity of M_i in the Jordan-Hölder series of N.

Proof (Read in book)

§ 9.3 Cartan Matrix

Let A be any finite-dim'l algebra with

$$\mathrm{Irr}\, A = \{M_1, \ldots, M_n\}$$

and projective covers P_i.

Let $c_{ij} = \dim \mathrm{Hom}_A(P_i, P_j)$

Def (c_{ij}) is the **Cartan matrix** of A.

Ex $A = \mathbb{C}(g,x) \Big/ \begin{pmatrix} g^2 = 1 \\ x^2 = 0 \\ gx + xg = 0 \end{pmatrix}$

$A = \mathbb{C}1 \oplus \mathbb{C}g \oplus \mathbb{C}x \oplus \mathbb{C}xg$

$\operatorname{Rad} A = (x)$

$A/\operatorname{Rad} \cong \mathbb{C}[C_2]$ ($\langle g \rangle$ cyclic order 2)

$M_1 = \mathbb{C}_{triv} \quad M_2 = \mathbb{C}_{sgn}$

$\operatorname{End} M_1 \oplus \operatorname{End} M_2 = \begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{bmatrix}$

$e_{11}^\circ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_{21}^\circ = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$e_1 := e_{11} = \frac{1}{2}(1-g), \quad e_2 := e_{21} = \frac{1}{2}(1+g)$

$P_1 = Ae_1 \qquad P_2 = Ae_2$ (check!)

$\operatorname{Hom}_A(P_1, P_2) \cong e_1 A e_2 =$

$= \mathbb{C}e_1 x e_2 + \mathbb{C}e_1 x g e_2$ ($ge_2 = e_2$)

$= \mathbb{C}x e_2^2 = \mathbb{C}x e_2 \quad \dim = 1$

Similarly $\mathrm{Hom}_A(P_2, P_1) \cong \mathbb{C} \times e_1$

Also $\mathrm{Hom}_A(P_i, P_i) =$

$= e_i A e_i = \mathbb{C} e_i$

So the cartan matrix is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

§9.4

Def Projective dimension of an A-module M (A = any ring), denoted $pd(M)$, is the shortest length of a projective resolution of M

$$pd(M) = \inf\left\{ d \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c}\text{there is a proj.} \\ \text{resol.} \\ 0 \to P_d \to \cdots \to P_0 \to M \to 0\end{array} \right\}$$

(so is ∞ if M has no finite proj. res.)

Def The homological dimension of A is d if every A-module M has $pd(M) \leq d$.

Cartan matrix of P_Q?

(Q = finite oriented graph without cycles)

Lecture 39

§ 9.5 Blocks

A fin. dim'l alg / $k = \bar{k}$

$\mathcal{C}$ = cat of f.d. A-modules

Def Two simple A-modules X, Y are linked if there is a chain

$$X = M_0, M_1, \ldots, M_n = Y$$

of A-modules such that $\forall i = 0, 1, \ldots, n-1$ either

$$\operatorname{Ext}^1(M_i, M_{i+1}) \neq 0$$

or

$$\operatorname{Ext}^1(M_{i+1}, M_i) \neq 0$$

(or both). We also agree any X is linked to itself.

Then being linked is an equivalence relation.

So the set of (isoclasses) $\mathrm{Irr}\, A$ of simple A-modules decomposes into equivalence classes

$$\mathrm{Irr}\, A = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n$$

The <u>k:th block</u> $\mathcal{C}_k$ of $\mathcal{C}_k$ is the category of all objects M in $\mathcal{C}$ such that all simple modules from the Jordan-Hölder series of M belongs to S_k.

Ex If A is semisimple, each S_k is a singleton

$$S_k = \{M_k\}$$

And each block C_k is equivalent to vect.

Ex If A is commutative local artinian, then there is only one block

$$\operatorname{Irr} A = S_1$$

(Eg. $A = k[x]/(x^m)$)

Lecture 40 May 2, 2025

Problem 953

i) indec central idempotents $\leftrightarrow$ blocks.

ii) $\mathcal{C} = \bigoplus_{k=1}^{n} \mathcal{C}_k$

Pf: Any indec M belongs to exactly one $\mathcal{C}_k$. And.

$\mathrm{Hom}(M,N) = 0$ $\quad M \in \mathcal{C}_k$, $N \in \mathcal{C}_\ell$, $k \neq \ell$

How to see this? Let $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ be a Composition series.

Say $n = 2$. Then

$0 \to M_1 \to M \to M/M_1 \to 0$

with M_1 and M/M_1 simple. If they are not linked, $M \cong M_1 \oplus (M/M_1)$ ∎

§ 9.6

$\mathcal{C}$ = k-linear abelian cat s.t. all objects have finite length, $\mathrm{End}(X) = k$ $\forall$ simple X.

Def $\mathcal{C}$ is finite if it has enough projectives and finitely many (up to iso) simple objects.

Def A progenerator $P \in \mathcal{C}$ is a proj. object st any $M \in \mathcal{C}$ is a quotient of $P^{\oplus n}$, some n.

Let $B = \mathrm{End}(P)^{op}$

Th. $F : \mathcal{C} \to {}_B\underline{Mod}$

$$X \longmapsto \mathrm{Hom}(P, X)$$

is an equivalence of categories.

Proof $G : {}_B\underline{Mod} \to \mathcal{C}$

$$G(X) := P \underset{B}{\otimes} X$$

i) $\forall X \in {}_B\underline{Mod}$, $X \simeq \mathrm{Hom}(P, P \underset{B}{\otimes} X)$

$$x \longmapsto (p \longmapsto p \otimes x)$$

Let $\varphi : P \to P \underset{B}{\otimes} X$

$$\begin{array}{ccccc} & & P & & \\ & \tilde{\varphi} \swarrow & \downarrow \varphi & & \\ P \otimes X & \to & P \underset{B}{\otimes} X & \to & 0 \end{array}$$

§9.7.

Def An alg A is **basic** if $A/\mathrm{Rad}A$ is commutative.

Th Every fd alg/$k=\bar{k}$ equivalent to a basic alg.

Pf $A/\mathrm{Rad}A \cong \bigoplus_{i=1}^{n} \mathrm{End}(V_i)$

Let $P = P_1 \oplus \cdots \oplus P_n$

By Exercise 9.6.3, P is a progenerator of the cat of fd A-modules.

Consider $B = \mathrm{End}(P)^{op} =$

$= \left[\bigoplus \mathrm{Hom}(P_i, P_j)\right]^{op}$

We claim B is basic. Can show

$B/\mathrm{Rad}\,B = \left[\bigoplus \mathrm{Hom}(V_i, V_j)\right]^{op}$

$= \bigoplus \mathrm{End}(V_i)^{op} = k \times \cdots \times k$ ∎

Lecture 43-44 May 9, 2025

The Jones Polynomial of a Knot (or Link)

① $U = U_q(sl_2)$

$$\begin{cases} [E,F] = \dfrac{K-K^{-1}}{q-q^{-1}} \\ KEK^{-1} = q^2 E \\ KFK^{-1} = q^{-2}F \end{cases}$$

② A 2-dim'l representation $V = \mathbb{C}v_{-1} \oplus \mathbb{C}v_{+1}$

$$E = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix} \quad K = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$$

What must α, β be? $\quad Kv_i = q^i v_i$

$$[E,F] = \begin{bmatrix} \alpha\beta & \\ & -\alpha\beta \end{bmatrix} = \frac{1}{q-q^{-1}} \begin{bmatrix} q-q^{-1} & 0 \\ 0 & q^{-1}-q \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

so $\alpha = \beta = 1$ works.

③ Comultiplication

$$\Delta(E) = E \otimes g + h \otimes E$$

What must g, h be?

$E = (\varepsilon \otimes 1)\Delta(E) = \varepsilon(h)E \Rightarrow \varepsilon(h) = 1$

similarly $\varepsilon(g) = 1$.

$$q^2\Delta(E) = \Delta(K)\Delta(E)\Delta(K)^{-1}$$
$$= (K \otimes K)(E \otimes K^{-1} + \cdots) \Rightarrow \begin{cases} u = \lambda K^m \\ v = \mu K^n \end{cases}$$

want

$$[\Delta(E), \Delta(F)] = \frac{\Delta(K) - \Delta(K)^{-1}}{q-q^{-1}} =$$

$$= \frac{1}{q-q^{-1}} (K \otimes K - K^{-1} \otimes K^{-1})$$

Have

$$\Delta(E)\Delta(F) = (E \otimes K^a + K^b \otimes E) \times$$
$$\times (F \otimes K^c + K^d \otimes F) =$$

$$= EF \otimes K^{a+c} + EK^d \otimes K^a F$$
$$+ K^b F \otimes EK^c + K^{b+d} \otimes EF$$

Switch E, F & subtract

$$[\Delta(E), \Delta(F)] = [E,F] \otimes K^{a+c} + K^{b+d} \otimes [E,F]$$
$$+ \cdots$$

The rest has to cancel. Leads to $a+c=1$, $b+d=-1$

$$\boxed{\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E \\ \Delta(F) &= F \otimes 1 + K^{-1} \otimes F \end{aligned}}$$

check:

$$[\Delta(E), \Delta(F)] = [EF] \otimes K + K^{-1} \otimes [E,F] +$$
$$+ EK^{-1} \otimes KF - K^{-1}E \otimes FK$$
$$+ [1 \otimes E, F \otimes 1] =$$

$$= \frac{1}{1-q^{-1}} (K \otimes K + K^{-1} \otimes K^{-1}) \quad \otimes K$$

④ Antipode

$$\Delta(E) = E \otimes K + 1 \otimes E$$
$$(S \otimes 1)\Delta(E) = S(E)K + 1 \cdot E$$

This must equal $\varepsilon(E) = 0$

$$\Rightarrow S(E) = -EK^{-1}$$

Similarly $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$

$$0 = (S \otimes 1)\Delta(F) = S(F) + S(K^{-1})F$$
$$\Rightarrow S(F) = -KF$$

(5) Invariant tensor

$$T = a \cdot v_{-1} \otimes v_1 + b\, v_1 \otimes v_{-1} \in V \otimes V$$

$$E.T = a\left[(E v_{-1}) \otimes K v_1 + v_{-1} \otimes E v_1\right] + b\left[(E v_1) \otimes K v_{-1} + v_1 \otimes E v_{-1}\right]$$

$$= (a + b \cdot q^{-1})\, v_{-1} \otimes v_{-1}$$

$$\Rightarrow \boxed{T = v_{-1} \otimes v_1 - q\, v_1 \otimes v_{-1}}$$

Check: $F.T = (F v_{-1}) \otimes v_1 + K^{-1} v_{-1} \otimes F v_1 - q\left[E v_1 \otimes v_{-1} + K^{-1} v_1 \otimes F v_{-1}\right]$

$$= v_1 \otimes v_1 - q q^{-1} v_1 \otimes v_1 = 0$$

Ok

Define coevaluation

$$coev : \mathbb{1} \longrightarrow V \otimes V$$

by $1_k \longmapsto T$

⑥ Invariant bilinear form

$B \in (V \otimes V)^*$

$B(v_1 \otimes v_1) = B(v_{-1} \otimes v_{-1}) = 0$

$\varepsilon(K^{-1}E)$

$0 = (K^{-1}E).B(v_{-1} \otimes v_{-1}) =$

$= B(E_{(1)} v_{-1} \otimes E_{(2)} v_{-1})$

$= B(E v_{-1} \otimes K v_{-1}) + B(v_{-1} \otimes E v_{-1})$

$= q^{-1} B(v_1 \otimes v_{-1}) + B(v_{-1} \otimes v_1)$

$\{\varepsilon S(K^{-1}E) = S(E)K = -E\}$

$E \begin{pmatrix} v_1 \\ v_{-1} \end{pmatrix} F$

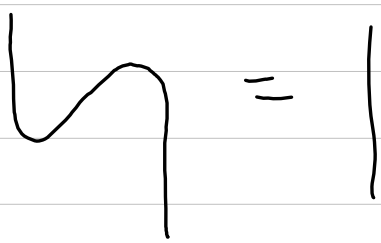
so $B(v_1 \otimes v_{-1}) = -q B(v_{-1} \otimes v_1)$

WLOG $\Rightarrow$

$B(v_{-1} \otimes v_1) = \cancel{1} \; -q^{-1}$

$B(v_1 \otimes v_{-1}) = \cancel{-q} \; 1$

$B(v_i \otimes v_i) = 0, \quad i = \pm 1$



$V \otimes 1$

$\downarrow 1 \otimes coev$

$V \otimes V \otimes V$

$\downarrow ev \otimes 1$

$1 \otimes V$

$v \mapsto v_i \otimes T = v_i \otimes (v_{-1} \otimes v_1 - q v_1 \otimes v_{-1})$

$\mapsto \delta_{i,1}(-q) v_1 - q \delta_{i,-1} v_{-1} = -q v_i$

⑦ R-matrix

For $U(\mathfrak{sl}_2)$, the flip

$$\tau_{V,W} : V \otimes W \longrightarrow W \otimes V$$

is an isomorphism. This fails for $U_q(\mathfrak{sl}_2)$ due to non-cocommutativity. $(\Delta \neq \Delta^{cop})$

$$\tau R = \begin{array}{c} \begin{matrix} 1 & 1 & 1,-1 & -1,1 & -1,-1 \end{matrix} \\ \begin{bmatrix} 1 & & & & \\ & a & b & & \\ & c & d & & \\ & & & & 1 \end{bmatrix} \end{array}$$

Want $\tau R : V \otimes V \to V \otimes V$ intertwining op.

Drop τ by abuse of notation.

$$E R\, v_i \otimes v_j = E\, v_i \otimes v_j = \begin{cases} 0 \\ v_1 \otimes q^{-1} v_{-1} + v_{-1} \otimes v_1 \end{cases}$$

$$R E\, v_i \otimes v_j = R\,(v_1 \otimes q^{-1} v_{-1} + v_{-1} \otimes v_1) \quad i=-1$$

$(i=-1)$

$$= q^{-1}(a\, v_1 \otimes v_{-1} + c\, v_{-1} \otimes v_1) + (b\, v_1 \otimes v_{-1} + d\, v_{-1} \otimes v_1)$$

$$= (q^{-1}a + b)\, v_1 \otimes v_{-1} + (q^{-1}c + d)\, v_{-1} \otimes v_1$$

$$\begin{cases} q^{-1}a + b = q^{-1} \\ q^{-1}c + d = 1 \end{cases}$$

$$E \begin{pmatrix} V_1 \\ \searrow \\ V_{-1} \end{pmatrix} F$$

$$E R\, v_1 \otimes v_{-1} = E(a\, v_1 \otimes v_{-1} + c\, v_{-1} \otimes v_1) = a\, v_1 \otimes v_1 + c q\, v_1 \otimes v_1$$

$$R E\, v_1 \otimes v_{-1} = R\, v_1 \otimes v_1 = v_1 \otimes v_1 \quad \Rightarrow a = 0 \quad c = q^{-1}$$

$\Delta(E) = E \otimes K + 1 \otimes E$

$$\begin{cases} R\, v_i \otimes v_i = v_i \otimes v_i \\ R\, v_+ \otimes v_- = a\, v_+ \otimes v_- + c\, v_- \otimes v_+ \\ R\, v_- \otimes v_+ = b\, v_+ \otimes v_- + d\, v_- \otimes v_+ \end{cases}$$

Want $\Delta(E)R = R\Delta(E)$ on $V = \mathbb{C}^2 = \mathbb{C}v_+ \oplus \mathbb{C}v_-$

We have $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad E(v_+) = v_-$, $E(v_-) = 0$

$$\Delta(E)R\, v_{--} = (E \otimes K + 1 \otimes E)(v_{--}) = q^{-1} v_{+-} + v_{-+}$$

$$R\Delta(E)\, v_{--} = R(q^{-1} v_{+-} + v_{-+}) =$$

$$= q^{-1}(a v_{+-} + c v_{-+}) + b v_{+-} + d v_{-+}$$

$$= (q^{-1}a + b)\, v_{+-} + (q^{-1}c + d)\, v_{-+}$$

We get $\begin{cases} q^{-1}a + b = q^{-1} \\ q^{-1}c + d = 1 \end{cases}$ $\quad \boxed{\begin{array}{l} a + qb = 1 \\ c + qd = q \end{array}}$ (1a) (1b)

$$\Delta(E) R\, v_{+-} = \Delta(E)(a v_{+-} + c v_{-+}) = (a + qc) v_{++}$$

$$R\Delta(E) v_{+-} = R(v_{++}) = v_{++} \qquad \boxed{a + qc = 1} \quad (2)$$

$$\Delta(E) R\, v_{-+} = \Delta(E)(b v_{+-} + d v_{-+}) = (b + qd)\, v_{++}$$

$$R\Delta(E) v_{-+} = R(q v_{++}) = q v_{++} \qquad \boxed{b + qd = q} \quad (3)$$

(2) $\Rightarrow$ $a = 1 - qc$

(3) $\Rightarrow$ $b = q - qd$

(1a) $\Rightarrow$ $(1 - qc) + q(q - qd) = 1$

$\Leftrightarrow$ $\quad q^2(1 - d) = qc$

$$q(1 - d) = c \qquad (4)$$

Subst (4) into (1b):

$q(1-d) + qd = q$ OK $\quad$ 1 free parameter

$$\boxed{\begin{aligned} a+qb &= 1 \quad (1a)\\ c+qd &= q \quad (1b)\end{aligned}}$$

Solve a, c, d in terms of b:

$$a \overset{(1a)}{=} 1-qb$$

$$\boxed{a+qc=1} \quad (2)$$

$$d \overset{(3)}{=} 1-q^{-1}b$$

$$c = q-qd = q-(q-b) = \underline{\underline{b}}$$

$$\boxed{b+qd=q} \quad (3)$$

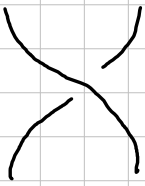
So
$$R = \begin{bmatrix} 1 & & & \\ & 1-qb & b & \\ & b & 1-q^{-1}b & \\ & & & 1 \end{bmatrix}$$

actual R-matrix

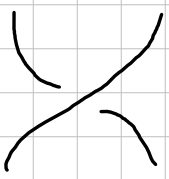
$$R = \tau \cdot \check{R}$$

$$\tau = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

Representing

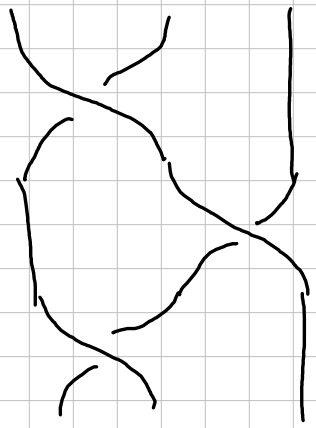


$= R$

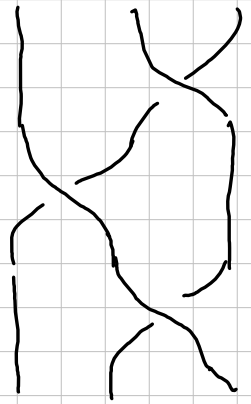


$= R^{-1}$

we want



$=$



$$R_{12} = R \otimes 1$$
$$R_{23} = 1 \otimes R$$

$$R_{12}R_{23}R_{12} =$$
$$= R_{23}R_{12}R_{23}$$

$$\Leftrightarrow b = 0, q, q^{-1}$$

With $b = q$ we get

$$R = \begin{pmatrix} 1 & & & \\ & 1-q^2 & q & \\ & q & 0 & \\ & & & 1 \end{pmatrix}$$

Note:
$\det R = -q^2$
$q^{-1}R$ has $\det = -1$

$$\begin{cases} R\, v_{+-} = (1-q^2)\, v_{+-} + q\, v_{-+} \\ R\, v_{-+} = q\, v_{+-} \end{cases}$$

$$\begin{cases} R^{-1} v_{+-} = q^{-1} v_{-+} \\ R^{-1} v_{-+} = (1-q^{-2})\, v_{-+} + q^{-1} v_{+-} \end{cases}$$

$$\begin{aligned} v_{+-} &= (1-q^2)\, R^{-1} v_{+-} + q\, R\, v_{-+} \\ &= (1-q^2)\, q^{-1} v_{-+} + q\, R\, v_{-+} \end{aligned}$$

$$q^{-1} v_{+-} + (1-q^{-2})\, v_{-+} = R\, v_{-+}$$

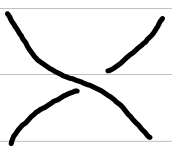
⑧ Graphical Calculus: Jones Polynomial

cap 

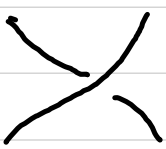
$\mathbb{1} \to V \otimes V$, $T = v_{-1} \otimes v_1 - q\, v_1 \otimes v_{-1}$

cup 

$V \otimes V \to \mathbb{1}$, B: $v_{-1} \otimes v_1 \mapsto 1$, $v_1 \otimes v_{-1} \mapsto -q$, $v_i \otimes v_i \mapsto 0$



$V \otimes V \to V \otimes V$, R: $v_{+-} \mapsto (1-q^2) v_{+-} + q\, v_{-+}$, $v_{-+} \mapsto q\, v_{+-}$, $v_{ii} \mapsto v_{ii}$



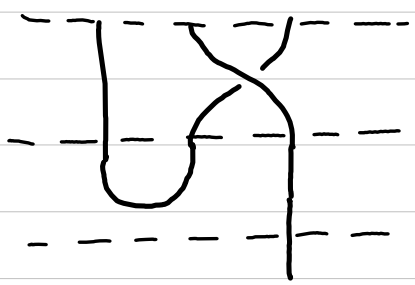
$V \otimes V \to V \otimes V$, R^{-1}: $v_{+-} \mapsto q^{-1} v_{-+}$, $v_{-+} \mapsto (1-q^{-2}) v_{-+} + q^{-1} v_{+-}$, $v_{ii} \mapsto v_{ii}$



$V \to V$, Id

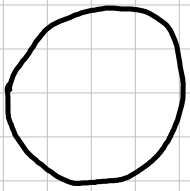


$V \otimes V \otimes V \to V \otimes V \otimes V$, $\mathrm{Id}^{\otimes 3}$



$V \otimes V \otimes V \to V \otimes V \otimes V \to V \otimes V \otimes V \to \mathbb{1} \otimes V \cong V$, $\mathrm{Id} \otimes R$, $B \otimes \mathrm{Id}$

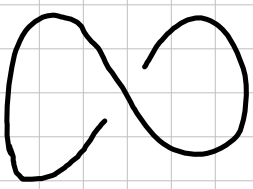
The Unknot



$$\mathbb{1} \xrightarrow{coev = T} V \otimes V \xrightarrow{ev = B} \mathbb{1}$$

$$ev(coev(1_k)) = ev(T)$$
$$= ev(V_{-+} - qV_{+-})$$
$$= 1 + q^2$$

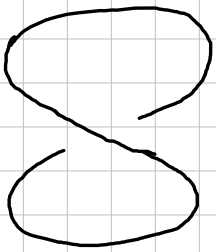
Try

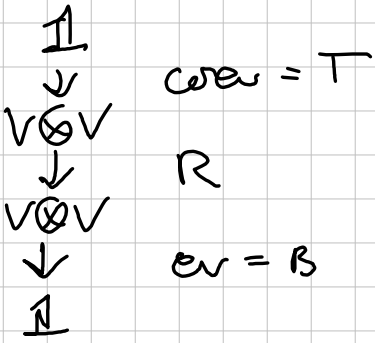


$$\mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1} \xrightarrow{coev \otimes coev} V \otimes V \otimes V \otimes V \xrightarrow{R_{23} = 1 \otimes R \otimes 1} V \otimes V \otimes V \otimes V \xrightarrow{ev \otimes ev} \mathbb{1} \otimes \mathbb{1}$$

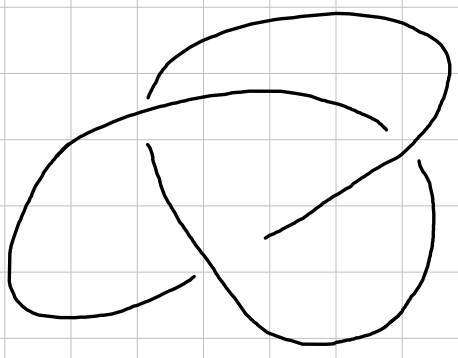
$T \otimes T$

$$(ev \otimes ev) \circ R_{23}((V_{-1} \otimes V_1 - qV_1 \otimes V_{-1}) \otimes (V_{-1} \otimes V_1 - qV_1 \otimes V_{-1})) =$$
$$= (ev \otimes ev) \circ R_{23}(V_{-+-+} - qV_{+--+} - qV_{-++-} + q^2 V_{+-+-}) =$$
$$= (ev \otimes ev) \circ (V_- \otimes ((1-q^2)V_{+-} + q^{-1}V_{-+}) \otimes V_+ - qV_{+--+} - qV_{-++-} + q^3 V_{++--})$$
$$= (1-q^2) + 0 + q^2 + q^2 + 0 = 1 + q^2$$

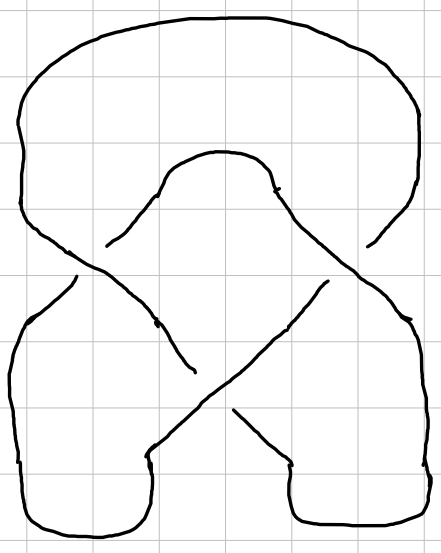




$$B(R\ (T))$$
$$\simeq B(R(v_{-+} - q v_{+-}))$$
$$= B(q v_{+-} - q((1-q^2)v_{+-} + q v_{-+}))$$
$$= B(q^3 v_{+-} - q^2 v_{-+})$$
$$= q^3(-q) - q^2$$
$$= -q^2(1+q^2)$$



Trefoil Knot



$\mathbb{1} \xrightarrow{T} V \otimes V \cong V \otimes \mathbb{1} \otimes V \xrightarrow{1 \otimes T \otimes 1} V \otimes V \otimes V \otimes V \xrightarrow{R \otimes R} V \otimes V \otimes V \otimes V \xrightarrow{1 \otimes R^{-1} \otimes 1} V \otimes V \otimes V \otimes V \xrightarrow{B \otimes B} \mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$

$$v_{-+} - q v_{+-} \simeq v_- \otimes 1 \otimes v_+ - q v_+ \otimes 1 \otimes v_-$$

$$\mapsto v_- \otimes (v_{-+} - q v_{+-}) \otimes v_+ - q v_+ \otimes (v_{-+} - q v_{+-}) \otimes v_-$$

$$= v_{--++} - q v_{-+-+} - q v_{+-+-} + q^2 v_{++--}$$

$R \otimes R$:

$$\mapsto v_{--++} - q^3 v_{+-+-} - q\left[(1-q^2) v_{+-} + q v_{-+}\right]^{\otimes 2} + q^2 v_{++--}$$

$$= v_{--++} - (q^3 - q(1-q^2)^2) v_{+-+-} - q^2(1-q^2) v_{+--+}$$
$$- q^2(1-q^2) v_{-++-} + q^2 v_{++--}$$

$1 \otimes R^{-1} \otimes 1$:

$$\mapsto v_- \otimes ((1-q^{-2}) v_{-+} + q^{-1} v_{+-}) \otimes v_+$$
$$- (q^3 - q(1-q^2)^2) \cdot v_+ \otimes ((1-q^{-2}) v_{-+} + q^{-1} v_{+-}) \otimes v_-$$

$$- q^2(1-q^2) v_{+--+} - q^2(1-q^2) v_{-++-}$$
$$+ q^2 q^{-1} v_{+-+-}$$

B: $-+ \mapsto 1$, $+- \mapsto -q$

$B \otimes B$:

$$\mapsto 0 + q^{-1} - (q^3 - q(1-q^2)^2)(-q)^2 (1-q^{-2})$$
$$- q^2(1-q^2)(-q) - q^2(1-q^2)(-q)$$
$$+ q(-q)^2 =$$

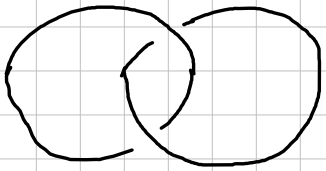
$$= q^{-1} - (q^3 - q(1-2q^2+q^4))(q^2-1)$$
$$+ 2(q^3 - q^5) + q^3$$

$$= q^{-1} - q^5 + q^3 + (q + 2q^3 + q^5) q^2 - (q + 2q^3 + q^5)$$
$$+ 3q^3 - 2q^5$$

$$= q^{-1} - 3q^5 + 4q^3 + q^3 + 2q^4 + q^7 - q - 2q^3 - q^5$$

$$= \boxed{q^{-1} - q + 3q^3 + 2q^4 - 4q^5 + q^7}$$

Exercise: Correct this!



$\mathbb{1}\otimes\mathbb{1}$

$\downarrow T\otimes T$

$V\otimes V\otimes V\otimes V$

$\downarrow 1\otimes R^2\otimes 1$

$V\otimes V\otimes V\otimes V$

$\downarrow B\otimes B$

$\mathbb{1}\otimes\mathbb{1}\simeq\mathbb{1}$

$1_k \mapsto 1_k\otimes 1_k \longmapsto (v_{-1}\otimes v_1 - q\, v_1\otimes v_{-1})^{\otimes 2} =$

$= v_{-+-+} - q\, v_{+--+} - q\, v_{-++-} + q^2 v_{+-+-}$

$\xmapsto{1\otimes R\otimes 1} v_-\otimes\big((1-q^2)v_{+-} + q\, v_{-+}\big)\otimes v_+ - q\, v_{+--+} - q\, v_{-++-}$

$+ q^2 v_{++--}$

$\xmapsto{1\otimes R\otimes 1} (1-q^2)\, v_-\otimes\big((1-q^2)v_{+-} + q\, v_{-+}\big)\otimes v_+$

$+ q^2 v_{-+-+} - q\, v_{+--+} - q\, v_{-++-}$

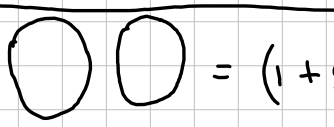
$+ q^2 v_+\otimes\big((1-q^2)v_{+-} - q\, v_{-+}\big)\otimes v_-$

$\xmapsto{B\otimes B} (1-q^2)^2 + q^2 + q^2 + q^2$

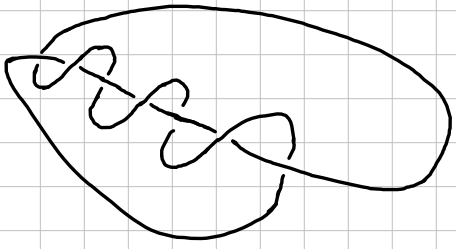
$+ q^2(-q)(-q)(-q) = \boxed{1 + q^2 + q^4 - q^5}$

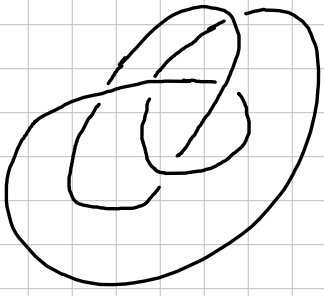
$-+ \xmapsto{B} 1$

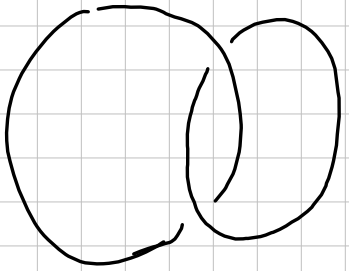
$+- \longmapsto -q$



$= (1+q^2)^2 = \boxed{1 + 2q^2 + q^4}$







$$v_{-+} - q v_{+-}$$

$$(v_{-+} - q v_{+-}) \otimes (v_{-+} - q v_{+-})$$

$$= v_{-+-+} - q v_{+--+} - q v_{-++-} + q^2 v_{+-+-}$$

$$\mapsto v_- \left((1-q^2) v_{+-} + q v_{-+} \right) v_+ - q v_{+--+}$$

$$- q v_{-++-} + q^3 v_{++--}$$

$$\cdots \boxed{q^{-2} + 1 + q^2 + q^4}$$